MAT235 Assignment 11 Solutions

5: Clearly the identity lies in the intersection of 2 subgroups. If a and b also lie in the intersection, then so does a^{-1} and ab since H_1 and H_2 are subgroups and hence closed under inversion and multiplication. Therefore so to is their intersection, hence $H_1 \cap H_2$ is a subgroup.

12: Since $< a >=< b >$, we must have $a = b^n$ and $b = a^m$ for some integers m and n. If a fixes some point $s \in S$, then clearly $bs = a^m s = s$ and so b also fixes s. Likewise a fixes s if b does. Hence $I(a) = I(b)$.

13: Recall that for $s \in S$ we have $|Orb(s)| = |G|/|Stab(s)|$. Let $s_1, \ldots, s_k$ be representatives of all the disjoint orbits in S. Then we have

$$N = \sum_{i=1}^{k} |Orb(s_i)| = \sum_{i=1}^{k} |G|/|Stab(s_i)|$$

Notice that every term in the sum on the right must be a power of p between 0 and r since p^r is the size of the group. It follows that if S has no fixed point (ie. no point for which $|Stab(s)| = |G|$, then every term in the sum is a positive power of p and not 1. But this would imply the entire sum, and hence N, is divisible by p, giving us a contradiction.

Symmetries of a cube: (i) Note that every symmetry sends a face to a face, of which there are 6 for a cube, and if every face is fixed, then the symmetry is trivial. Since for every pair of faces, it is obvious that there is some permutation that sends one face to another, the orbit of any face has 6 elements in it and hence it suffices to find the number of symmetries that fix some given face (by lemma 30.2.1 in the notes, $|G| = |Orb(s)||Stab(s)|$). Each face is a square, and there are 8 symmetries of a square (4 rotations and 4 reflections, the D_4 group). It is easy to see that each symmetry of a square face can be extended to a unique symmetry of the cube; any rotation can be extended to a rotation about an axis through the cube by the same degree, and any reflection can be extended to a reflection through the unique plane that contains the line of reflection and is perpendicular to the face. It follows there are $6 \times 8 = 48$ symmetries for the cube. Note that another approach to this problem is to consider vertices instead of faces.

(ii) There are no symmetries of order 5 because 5 does not divide 48. Again considering our group as a subset of S_6 (by its faithful action on the faces), we can consider the symmetry σ that flips the cube through a plane that runs parallel to two faces, then $\sigma = (16)$ for a certain labeling of the faces where 1 and 6, 2 and 5, 3 and 4 are opposite faces. We can also consider τ as the symmetry that rotates around an axis collinear with the diagonal of some pair of opposite vertices. Again $\tau = (123)(456)$ for the same labeling. Composing $\sigma\tau = (123645)$ is an element of order 6.

(iii) There are a number of ways to show that there are 24 rotations. One way is to recall Lagrange's theorem, ie. that the order of a subgroup must divide the order of the group. Since it is clear that not every symmetry is a rotation (ie. not orientation preserving) it is enough to construct 17 explicit rotations (48 /3 = 16) to prove that there must be 24 rotations total.

(iv) Realization as a subgroup of S_6 is related to the solution to (i), where it was noted that each permutation acts on the faces. It is clear that if a symmetry fixes every face, then it fixes the cube and must be the identity. By labeling the faces 1 through 6, we can construct an explicit injective homomorphism from G to S_6 by considering each element of G as a permutation by how it permutes the numbers corresponding to the faces (it is an injection because if a symmetry fixes every face, it must be the identity). There is no subgroup of order 48 in S_5 (Lagrange). To realize H as a subgroup of S_4, consider the action of H on the diagonals connecting the opposite vertices of the cube (there are 4 such diagonals so label them 1 through 4). It is easy to show that there are rotations (ie. elements of H) that act as 3 cycles and rotations that act as 4 cycles on these diagonals if we proceed as above in realizing these rotations as permutations. Argue that all such elements generate the whole group S_4, which has the same order as H.

1 (i) We use the CFF with the group $\mathbb{Z}/12\mathbb{Z}$ and S the number of all possible painted roulettes (so $|S| = \binom{12}{7} \times \binom{5}{2} = 792 \times 10 = 7920$). Note that from problem 13 above, if $< a >=< b >$ then $I(a) = I(b)$ so it is enough to find $I(a)$ for $a \in \{\overline{0}, \overline{1}, \overline{2}, \overline{3}, \overline{4}, \overline{6}\}$, where $I(a)$ is the number of elements of S fixed by the action of a. Clearly $I(\overline{0}) = |S| = 7920$. Clearly $I(\overline{1}) = 0$ because for i to fix a roulette, every wedge would have to be the same. By the above remark, it follows that $I(\overline{5}) = I(\overline{7}) = I(\overline{11})$ are also all 0. Likewise $I(\overline{2}) = I(\overline{10})$ must be zero because otherwise there would be at most 2 colours. For 3 to fix an element, every colour appearing would have to be divisibly by 4, and similarly for 4 or 6 to fix an element, every colour appearing would have to be divisibly by 3 or 2 respectively. Since this is not the case, we get that $I(\overline{n}) = 0$ for all n except 0. Hence the number of orbits by the CFF is $7920/12 = 660$.

(ii) We proceed similarly as above, except now the group of interest is D_{12}. Again $I(e) = |S| = 7920$ and $I(y^n) = 0$ for $1 \leq n \leq 11$ because this action is the same as the action calculated for n above. It remains to consider xy^n for $0 \leq n \leq 11$, where say x is flipping through the axis that intersects stone 12 and stone 6 (so $x =$(1 11)(2 10)(3 9)(4 8)(5 7)). Notice that xy^n for n even are all flipping through axes that intersect two stones, (ie. 5 disjoint transpositions) and xy^n for n odd are all flipping through axis that lie between the stones (ie. 6 disjoint transpositions). It follows that $I(xy^n) = 0$ for n odd because for such an element to fix a necklace, every colour would have to appear an even number of times. For x to fix a necklace, we must have that stones 12 and 6 must be either blue or green, with each colour appearing once (this is again because of a parity argument like the one above). The stones from 1 through 5 must consist of 1 red, 1 blue and 3 green, and these choices determine a unique arrangement of stones fixed by x. Hence $I(x) = 2 \times \binom{5}{1} \times \binom{4}{1} = 40$. Since $I(xy^n) = I(x)$ for all n even (there is an obvious bijection between the elements that they fix), we have

$$N = 1/24 \times (7920 + 40 \times 6) = 340.$$

2 (i) Again $\mathbb{Z}/12\mathbb{Z}$ acts on the set S of all possible colourings of roulettes and $|S| = \binom{12}{2} \times \binom{10}{4} = 13860$. As before $I(\overline{0}) = 13860$, $I(\overline{1}) = I(\overline{5}) = I(\overline{7}) = I(\overline{11}) = 0$, $I(\overline{2}) = I(\overline{10}) = 0$, $I(\overline{3}) = I(\overline{9}) = 0$ and $I(\overline{4}) = I(\overline{8}) = 0$. Finally, $\overline{6}$ as an action on 12 elements looks like 6 disjoint transpositions and so we can consider each transposition as a pair that must share the same colour and for which 1 pair must be red, 2 pairs blue and 3 pairs green. Thus there are $\binom{6}{1} \times \binom{5}{2} = 60$ possible pairings. Hence the number of distinct roulettes (equivalently the number of orbits of S under the above action) is

$$N = 1/12 \times (13860 + 60) = 1160.$$

2 (ii) As before we have $I(e) = 13860$, $I(y^6) = 60$ and $I(y^n) = 0$ for $n \neq 0, 6$. We are again left with considering the two different cases of flipping. For xy^n with n even and x as above, we have 5 disjoint transpositions. There are 3 subcases, depending on the colouring of the stones that the reflection axis passes through. Case 1 is that these two stones are coloured red, then we can colour the remaining pairs in $\binom{5}{2} = 10$ ways. Case 2 is that these stones are blue, then there are $\binom{5}{1} \times \binom{4}{1} = 20$ choices. Finally if the two stones on the axis are green, we have $\binom{5}{1}\binom{4}{2} = 30$ choices. In total there are $10 + 20 + 30 = 60$ arrangements that are fixed by xy^n for each even n. For xy^n and n odd, we have 6 disjoint transpositions corresponding to 6 pairs that must have the same colouring. Hence the number of elements fixed by such an xy^n is $\binom{6}{1}\binom{5}{2} = 60$. We conclude

$$N = 1/24(13860 + 6 \times 60 + 6 \times 60 + 60) = 610.$$

3 (i) Similarly as above, except now we have $\mathbb{Z}/14\mathbb{Z}$ and the order of S is $\binom{14}{2}\binom{12}{4}\binom{8}{3} = 2522520$. As in (1), the common divisor of the number of stones is 1, and we can conclude that only the trivial group element fixes anything in S. It follows that

$$N = 1/14 \times 2522520 = 180180.$$

3 (ii) We are now considering the group action of D_{14}, a group of order 28. As before, the previous result applies so that $I(y^n)$ is 0 unless $n = 0$ in which case $I(y^n) = |S|$. Assume x is a reflection about an axis running through 14 and 7, so is represented in S_{14} as 6 disjoint transpositions. There are two colours that appear an odd number of times, and so as in (1), $I(xy^n) = 0$ if n is odd. If n is even, we must have both green and black appearing as colours of the two stones intersecting the axis of reflection, and for the other 6 pairs of stones, we have $\binom{6}{1}\binom{5}{2}\binom{3}{1} = 180$ choices. Hence $I(xy^n) = 2 \times 180 = 360$ for each n even. We conclude

$$N = 1/28(2522520 + 7 \times 360) = 90180.$$