Math 235: Assignment 1 Solutions

1.1: For \( n \in \mathbb{N} \) not zero, let \( A_n = [-n, n] \) (The closed interval in \( \mathbb{R} \) containing all real numbers \( x \) satisfying \( -n \leq x \leq n \)). It is easy to see that we have the chain of inclusion \( A_1 \subset A_2 \subset A_3 \subset \ldots \), and therefore

\[
\bigcup_{n=1}^{N} A_n = A_N = [-N, N]
\]

and \( \bigcap_{n=1}^{N} A_n = A_1 = [-1, 1] \).

1.2: Notice that the consecutive closed intervals, \([n, n+1]\) and \([n+1, n+2]\) do in fact overlap (This is because we have taken the closed interval. What if we had taken the open intervals instead?) We conclude that the finite union is

\[
\bigcup_{n=1}^{N} [n, n+1] = [1, N+1].
\]

Since any real number \( x \geq 1 \) will eventually lie inside the interval \([1, N+1]\) for some sufficiently large \( N \in \mathbb{N} \), we conclude

\[
\bigcup_{n=1}^{\infty} [n, n+1] = [1, \infty).
\]

(Note we have the open bracket “)” on the right side of the above interval. Why?). Similarly, the consecutive open intervals \((n, n+2)\) and \((n+1, n+3)\) also overlap, so

\[
\bigcup_{n=1}^{N} (n, n+2) = (1, N+2)
\]

\[
\therefore \bigcup_{n=1}^{\infty} (n, n+2) = (1, \infty).
\]

(\( \therefore \) means "therefore")

1.3: It is helpful to say in words what each \( A_n \) is. \( A_1 \) is the set of all natural numbers, \( \mathbb{N} \). \( A_2 \) is the set of all squares of natural numbers. \( A_3 \) is all cubes, etc. Of course, a square of a natural number is in particular a natural number, so \( A_2 \subseteq A_1 \). Likewise for cubes, \( 4^{th} \) powers, etc. We conclude \( A_1 \supseteq A_n \) for all \( n \in \mathbb{N} \),

\[
\therefore \bigcup_{n=1}^{\infty} A_n = A_1 = \mathbb{N}.
\]

Again it is helpful to ask in words, what does \( \bigcap_{n=1}^{\infty} A_n \) mean? Since each \( A_n \) is the subset of natural numbers consisting of \( n^{th} \) powers, their intersection must be all natural numbers \( x \) such that \( x \) is a square, a cube, a \( 4^{th} \) power, a \( 5^{th} \), etc. There are only 2 such numbers:
\[
\bigcap_{n=1}^{\infty} A_n = \{0, 1\}.
\]

(Challenge: Can you find the smallest natural number \(x \neq 0, 1\) such that \(x\) is a square, a cube, a \(4^{th}\) power, ..., and a \(10^{th}\) power? What if we replace 10 with some arbitrarily large \(N^{th}\) power?)

2.1: Consider the sets \(A = B = C = \{1\}\). Then \((A \setminus B) \setminus C\) is the empty set \(\emptyset\), while \(A \setminus (B \setminus C) = A \setminus \emptyset = A\). This counterexample is enough to show that the formula in 2.1 does not hold (There are of course many more interesting counterexamples that show this as well, but any one is sufficient).

2.2: Note that no amount of examples will suffice to prove that this formula does hold, hence we need to get down to the nitty-gritty and construct a proof. In general, when proving two sets are equal (say \(X\) and \(Y\)), a good place to start is to try to show that if \(x \in X\) then \(x\) must also be in \(Y\) and vice versa. This will show that \(X\) is a subset of \(Y\) and \(Y\) is a subset of \(X\), which is possible only if \(X = Y\). We proceed in this manner:

Assume \(x \in (B \setminus A) \cup (A \setminus B)\).

\(\Rightarrow\) either \(x\) is in \(B \setminus A\) or \(x \in A \setminus B\) (Note that the word 'or' in mathematics includes the possibility of both events occurring unless explicitly stated otherwise, although in this case that is in fact not possible (Why?). Also \(\Rightarrow\) means 'this implies')

Case 1: \(x \in B \setminus A\).

Then \(x \in B\) but \(x \notin A\). \(\therefore x \in A \cup B\) but \(x \notin A \cap B\) \(\Rightarrow x \in (A \cup B) \setminus (A \cap B)\).

Case 2: \(x \in A \setminus B\).

Repeat the exact same argument as case 1, but reverse the letters \(A\) and \(B\). (This is because the equation is 'symmetric' with respect to the sets \(A\) and \(B\).

Hence in either case, \(x \in (A \cup B) \setminus (A \cap B)\). Since \(x\) was an arbitrary element of \((B \setminus A) \cup (A \setminus B)\), we conclude that \((A \cup B) \setminus (A \cap B) \supseteq (B \setminus A) \cup (A \setminus B)\) (Make sure you understand why). It remains to show the reverse inclusion.

Assume \(y \in (A \cup B) \setminus (A \cap B)\).

\(\Rightarrow y \in A \cup B\) but \(y \notin A \cap B\). Hence \(y\) is in either \(A\) or \(B\), but not both for in that case, \(y\) would also be in their intersection (Again make sure you understand this conclusion).

Case 1: \(y \in A\) and \(y \notin B\).

\(\Rightarrow y \in A \setminus B\)

\(\Rightarrow y \in (B \setminus A) \cup (A \setminus B)\).

Case 2: \(y \in B\) and \(y \notin A\).

Again by the symmetry in the argument we can follow case 1, just switching \(A\) and \(B\). We conclude \((B \setminus A) \cup (A \setminus B) \supseteq (A \cup B) \setminus (A \cap B)\). Since we have now proved both inclusions, we can conclude \((B \setminus A) \cup (A \setminus B) = (A \cup B) \setminus (A \cap B)\).

3: We assume the P.O.I. and prove the statement "every non-empty set of natural numbers contains a minimal element". Actually, we will prove the contrapositive of this statement, namely "Every set of natural numbers without a minimal element is necessarily empty" (Note that these two claims are logically equivalent, that is to say one statement is true if and only if the other is. Proof by contrapositive is
a common tactic in mathematics, so it is important that you understand what it means and how it works). Let \( X \subset \mathbb{N} \) be a subset that does not have a minimal element, and consider the following statement about the natural numbers:

\[
P(n) : \text{No natural number less then or equal to } n \text{ lies in } X.
\]

We would like to prove this statement is true for all natural numbers. We will proceed by induction.

Base case: \( n = 0 \).

\( P(0) \) says, albeit in a rather long winded fashion, that \( 0 \notin X \). Could \( 0 \) lie in \( X \)? Well if it did, then it would have to be the minimal element, since 0 is the smallest natural number. Since \( X \) has no minimal element, we conclude that \( 0 \notin X \) and therefore \( P(0) \) is true.

Induction step: Assume that \( P(k) \) is true for some \( k \in \mathbb{N} \). Does it follow that \( P(k+1) \) is also true?

\( P(k+1) \) says that no natural number less then or equal to \( k+1 \) lies in \( X \). By assumption, \( P(k) \) is true, and therefore no natural number less then or equal to \( k \) lies in \( X \). Hence \( P(k+1) \) is false if and only if \( k+1 \in X \). Again, we wonder if this is possible. If \( k+1 \in X \) and no natural number strictly less then \( k \) lies in \( X \) (Because we are assuming \( P(k) \) is true), then we must conclude that \( k+1 \) would be the minimal element of \( X \). Since \( X \) has no minimal element, we conclude this is not the case, therefore \( P(k+1) \) is true.

By the Principle of Induction, we conclude that \( P(n) \) holds for all \( n \in \mathbb{N} \). We claim that this shows \( X \) is necessarily empty. For if \( X \) was non-empty, then there would be some \( m \in \mathbb{N} \) such that \( m \in X \). But then \( P(m) \) would be false, contradictory to what we just proved (Make sure you understand why here). Hence \( X \) is empty, so we have proved the contrapositive to our original claim, which as mentioned earlier, proves our original statement is also true.

4:

Base case: \( n = 1 \).

The LHS of the equation is \( 1^3 \). The RHS is \( 1^2 \). Clearly equality holds in this case. (LHS and RHS are shorthand for 'left hand side' and 'right hand side')

Induction step: Assume our equation holds for some \( n = k \). Does this imply it holds for \( n = k+1 \)?

Working with the LHS of our equation, we have

\[
1^3 + 2^3 + \ldots + k^3 + (k+1)^3 = (1^3 + 2^3 + \ldots + k^3) + (k+1)^3
\]

\[
= (1 + 2 + \ldots + k)^2 + (k+1)^3,
\]

where the last inequality comes from our inductive assumption. Using the equality

\[
1 + 2 + \ldots + k = \frac{(k)(k+1)}{2}
\]

from the notes and then factoring, we get

\[
(1 + 2 + \ldots + k)^2 + (k+1)^3 = \frac{(k)^2(k+1)^2}{4} + (k+1)^3.
\]
\[
\frac{(k+1)^2(k^2+4k+4)}{4} = \frac{(k+1)^2(k+2)^2}{4}.
\]

Once again we use the formula from the notes and finally conclude that

\[
\frac{(k+1)^2(k+2)^2}{4} = \left(\frac{(k+1)(k+2)}{2}\right)^2 = (1 + 2 + \ldots + k + (k+1))^2.
\]

This proves the induction step and hence the formula holds for all \(n \in \mathbb{N}\).

5.1: Many different surjective functions exist, we list 4 below in set notation:

\[
\begin{align*}
    f_1 &:= \{(1,a), (2,b), (3,b), (4,c)\} \\
    f_2 &:= \{(1,a), (2,a), (3,b), (4,c)\} \\
    f_3 &:= \{(1,c), (2,a), (3,c), (4,b)\} \\
    f_4 &:= \{(1,b), (2,c), (3,c), (4,a)\}.
\end{align*}
\]

Note that the set notation for a function means, for example \(f_1(1) = a, f_1(2) = b\), etc.

5.2: Again, there are more than 4 injective functions from \(B\) to \(A\), but here are 4:

\[
\begin{align*}
    g_1 &:= \{(a,1), (b,2), (c,3)\} \\
    g_2 &:= \{(a,1), (b,3), (c,2)\} \\
    g_3 &:= \{(a,4), (b,3), (c,1)\} \\
    g_4 &:= \{(a,3), (b,1), (c,4)\}.
\end{align*}
\]

5.3: Recall that a function sends each element of its domain (In this case \(A\)) to a unique element in its range (In this case \(B\)). Since \(B\) has precisely 3 elements, that means when constructing a function \(f : A \rightarrow B\), for every \(x \in A\) there are 3 possibilities for \(f(x)\). Since \(A\) has 4 elements, we conclude there are \(3 \times 3 \times 3 \times 3\) possible combinations, or 81 different functions from \(A\) to \(B\).

5.4: A surjective function must hit every element in its range at least once. Since \(|A| = 4\) and \(|B| = 3\) we conclude that any surjective function hits one element of \(B\) twice, and the other two exactly once. There are 3 ways to choose the element that will be hit twice, so it suffices to find the number of functions that hit say \(a\) twice and the other two elements once, and then multiply that number by 3. How many functions is this? Well there are \(\binom{4}{2} = 6\) different ways to choose 2 elements of \(A\) that will hit \(a\), and once we have made that choice, there is still the question of where to send the remaining two elements. But we know there are only 2 possibilities for that choice, since \(b\) and \(c\) both need to be hit exactly once. Hence there are \(6 \times 2 = 12\) different surjective functions that hit \(a\) twice, and therefore \(12 \times 3 = 36\) different surjective functions altogether.

5.5: Suppose \(f : A \rightarrow B\) is an injective function. Injective means that each element of \(B\) is the \(f\)-image of at most one element of \(A\). Hence the set \(f^{-1}(B)\) has at most 3 elements total (Since in particular it has size at most equal to \(B\)). But \(B\) is the range space of \(f\), so \(f^{-1}(B) = A\) which of course has 4 elements. This is a contradiction and thus no such \(f\) exists.
5.6: Following the argument in 5.3, for each element $y$ of $B$, there are 4 possible images, and each function send $y$ to exactly one of them. Hence there are $4 \times 4 \times 4 = 64$ different functions.

5.7: We argue similarly to 5.5 to show there does not exist a surjective function $f : B \to A$. Indeed assuming otherwise, then since $f$ is surjective, every element of $A$ is in its image so in particular, the set $f^{-1}(A)$ has at least 4 distinct elements (Namely, one element that gets sent to 1, one that gets sent to 2, etc. Why must these elements be distinct?) But $f^{-1}(A) = B$ which has only 3 elements and $\therefore$ no such $f$ exists.

5.8: Similarly to 5.4, we reason combinatorially. Suppose $f : B \to A$ is some injective function. There are 4 possible place $f$ could send $a$. Once we have made the choice of where to send $a$, there are now 3 possible choices of where to send $b$ (Because $f(a) \neq f(b)$). Likewise, after choosing where to send $a$ and $b$, there are now only 2 possible choice for the image of $c$. Hence there are $4 \times 3 \times 2 = 24$ total possibile injective functions.