# ASSIGNMENT 5 - MATH235, FALL 2009

**Submit by 16:00, Tuesday, October 13 (use the designated mailbox in Burnside Hall,** $10^{th}$ **floor).**

1. Given an integer $N$ we write $N$ in decimal expansion as $N = n_k n_{k-1} \dots n_0$, the $n_i$ being the digits of $N$. Note that this means that $N = n_0 + 10n_1 + 10^2 n_2 + \dots + 10^k n_k$. In the following you are asked to show certain divisibility criteria that can be proved by using congruences.

(1) Prove that a positive integer $N = n_k n_{k-1} \dots n_0$ is divisible by 3 if and only if the sum of its digits $n_0 + n_1 + \dots + n_k$ is divisible by 3. (Hint: show that in fact $N$ and $n_0 + n_1 + \dots + n_k$ are congruent to the same number modulo 3.) Example: 34515 is divisible by 3 because $3 + 4 + 5 + 1 + 5 = 18$ is divisible by 3.

(2) Prove that a positive integer $N = n_k n_{k-1} \dots n_0$ is divisible by 11 if and only if the sum of its digits with alternating signs $n_0 - n_1 + n_2 - \dots \pm n_k$ is divisible by 11. The same Hint applies here. Example: 1234563 is divisible by 11 since $1 - 2 + 3 - 4 + 5 - 6 + 3 = 0$ is divisible by 11.

(3) Prove that a positive integer $N = n_k n_{k-1} \dots n_0$ is divisible by 7 if and only if when we let $M = n_k n_{k-1} \dots n_1$, we have $M - 2n_0$ is divisible by 7. Example: take the number $7 * 11 * 13 * 17 = 17017$. It is clearly divisible by 7. Let us check the criterion against this example. We form the number $1701 - 2 * 7 = 1687$ and then the number $168 - 2 * 7 = 154$ and then the number $15 - 2 * 4 = 7$. So it works. Let us also check the number 82. It is not divisible by 7, in fact it's residue modulo 7 is 5. Also $8 - 2 * 2 = 4$, so the criterion shows that it's not divisible. Note though that in this case the number $N = 82$ and the number $M = 8 - 2 * 2 = 4$ don't have the same residue modulo 7. So you need to construct your argument a little differently.

2. To check if you had multiplied correctly two large numbers $A$ and $B$, $A \times B = C$, you can make the following check: sum the digits of $A$; keep doing it repeatedly until you get a single digit number $a$. Do the same for $B$ and $C$ and get numbers $b, c$. If you have multiplied correctly, the sum of digits of $ab$ is $c$. Prove that this is so. This is called in French "preuve par neuf".

Example: I have multiplied $A = 367542$ by $B = 687653$ and got $C = 252741358926$. To check (though this doesn't prove the mulitplication is correct) I do: $3 + 6 + 7 + 5 + 4 + 2 = 27, 2 + 7 = 9$ and $a = 9$. Also $6 + 8 + 7 + 6 + 5 + 3 = 35, 3 + 5 = 8$ and $b = 8$. $ab = 72$ and its sum of digits is 9. On the other hand $2 + 5 + 2 + 7 + 4 + 1 + 3 + 5 + 8 + 9 + 2 + 6 = 54, 5 + 4 = 9$. So it checks.

3.

(1) Solve that equation $x^2 + x = 0$ in $\mathbb{Z}/5\mathbb{Z}$.
(2) Solve that equation $x^2 + x = 0$ in $\mathbb{Z}/6\mathbb{Z}$.
(3) Solve that equation $x^2 + x = 0$ in $\mathbb{Z}/p\mathbb{Z}$, where $p$ is prime.

4. Solve each of the following equations:
   (1) $12x = 2$ in $\mathbb{Z}/19\mathbb{Z}$.
   (2) $7x = 2$ in $\mathbb{Z}/24\mathbb{Z}$.
   (3) $31x = 1$ in $\mathbb{Z}/50\mathbb{Z}$.
   (4) $34x = 1$ in $\mathbb{Z}/97\mathbb{Z}$.
   (5) $27x = 2$ in $\mathbb{Z}/40\mathbb{Z}$.
   (6) $15x = 5$ in $\mathbb{Z}/63\mathbb{Z}$.

5.
   (1) Let $p > 2$ be a prime. Prove that an equation of the form $ax^2 + bx + c$ (where $a, b, c \in \mathbb{F}_p$, $a \neq 0$) has a solution in $\mathbb{Z}/p\mathbb{Z}$ if and only if $b^2 - 4ac$ is a square in $\mathbb{Z}/p\mathbb{Z}$. If this is so, prove that the solutions are given by the familiar formula.
   (2) Determine for which values of $a$ the equation $x^2 + x + a$ has a solution in $\mathbb{Z}/7\mathbb{Z}$.