ASSIGNMENT 5 - MATH235, FALL 2009

Submit by 16:00, Tuesday, October 13 (use the designated mailbox in Burnside Hall, 10th floor).

1. Given an integer \( N \) we write \( N \) in decimal expansion as \( N = n_k n_{k-1} \ldots n_0 \), the \( n_i \) being the digits of \( N \). Note that this means that \( N = n_0 + 10n_1 + 10^2n_2 + \cdots + 10^k n_k \). In the following you are asked to show certain divisibility criteria that can be proved by using congruences.

   (1) Prove that a positive integer \( N = n_k n_{k-1} \ldots n_0 \) is divisible by 3 if and only if the sum of its digits \( n_0 + n_1 + \cdots + n_k \) is divisible by 3. (Hint: show that in fact \( N \) and \( n_0 + n_1 + \cdots + n_k \) are congruent to the same number modulo 3.) Example: 34515 is divisible by 3 because \( 3 + 4 + 5 + 1 + 5 = 18 \) is divisible by 3.

   (2) Prove that a positive integer \( N = n_k n_{k-1} \ldots n_0 \) is divisible by 11 if and only if the sum of its digits with alternating signs \( n_0 - n_1 + n_2 - \cdots \pm n_k \) is divisible by 11. The same Hint applies here. Example: 1234563 is divisible by 11 since \( 1 - 2 + 3 - 4 + 5 - 6 + 3 = 0 \) is divisible by 11.

   (3) Prove that a positive integer \( N = n_k n_{k-1} \ldots n_0 \) is divisible by 7 if and only if when we let \( M = n_k n_{k-1} \ldots n_1 \), we have \( M - 2n_0 \) is divisible by 7. Example: take the number \( 7 \times 11 \times 13 \times 17 = 17017 \). It is clearly divisible by 7. Let us check the criterion against this example. We form the number \( 1701 - 2 \times 7 = 1687 \) and then the number \( 168 - 2 \times 7 = 154 \) and then the number \( 15 - 2 \times 4 = 7 \). So it works. Let us also check the number 82. It is not divisible by 7, in fact it’s residue modulo 7 is 5. Also \( 8 - 2 \times 2 = 4 \), so the criterion shows that it’s not divisible. Note though that in this case the number \( N = 82 \) and the number \( M = 8 - 2 \times 2 = 4 \) don’t have the same residue modulo 7. So you need to construct your argument a little differently.

2. To check if you had multiplied correctly two large numbers \( A \) and \( B \), \( A \times B = C \), you can make the following check: sum the digits of \( A \); keep doing it repeatedly until you get a single digit number \( a \). Do the same for \( B \) and \( C \) and get numbers \( b, c \). If you have multiplied correctly, the sum of digits of \( ab \) is \( c \). Prove that this is so. This is called in French “preuve par neuf”.

Example: I have multiplied \( A = 367542 \) by \( B = 687653 \) and got \( C = 252741358926 \). To check (though this doesn’t prove the multiplication is correct) I do: \( 3 + 6 + 7 + 5 + 4 + 2 = 27, 2 + 7 = 9 \) and \( a = 9 \). Also \( 6 + 8 + 7 + 6 + 5 + 3 = 35, 3 + 5 = 8 \) and \( b = 8 \). \( ab = 72 \) and its sum of digits is 9. On the other hand \( 2 + 5 + 2 + 7 + 4 + 1 + 3 + 5 + 8 + 9 + 2 + 6 = 54, 5 + 4 = 9 \). So it checks.

3.

   (1) Solve that equation \( x^2 + x = 0 \) in \( \mathbb{Z}/5\mathbb{Z} \).

   (2) Solve that equation \( x^2 + x = 0 \) in \( \mathbb{Z}/6\mathbb{Z} \).

   (3) Solve that equation \( x^2 + x = 0 \) in \( \mathbb{Z}/p\mathbb{Z} \), where \( p \) is prime.
4. Solve each of the following equations:
   (1) $12x = 2$ in $\mathbb{Z}/19\mathbb{Z}$.
   (2) $7x = 2$ in $\mathbb{Z}/24\mathbb{Z}$.
   (3) $31x = 1$ in $\mathbb{Z}/50\mathbb{Z}$.
   (4) $34x = 1$ in $\mathbb{Z}/97\mathbb{Z}$.
   (5) $27x = 2$ in $\mathbb{Z}/40\mathbb{Z}$.
   (6) $15x = 5$ in $\mathbb{Z}/63\mathbb{Z}$.

5. 
   (1) Let $p > 2$ be a prime. Prove that an equation of the form $ax^2 + bx + c$ (where $a, b, c \in \mathbb{F}_p$, $a \neq 0$) has a solution in $\mathbb{Z}/p\mathbb{Z}$ if and only if $b^2 - 4ac$ is a square in $\mathbb{Z}/p\mathbb{Z}$. If this is so, prove that the solutions are given by the familiar formula.
   (2) Determine for which values of $a$ the equation $x^2 + x + a$ has a solution in $\mathbb{Z}/7\mathbb{Z}$. 