ASSIGNMENT 4 - MATH235, FALL 2009

Submit by 16:00, Monday, October 5 (use the designated mailbox in Burnside Hall, 10^{th} **floor).**

1. The least common multiple of nonzero integers a, b is the smallest positive integer m such that $a|m$ and $b|m$. We denote it by $\mathrm{lcm}(a, b)$ or $[a, b]$. Prove that:
   (1) If $a|k$ and $b|k$ then $[a, b]|k$.
   (2) $[a, b] = \frac{ab}{(a,b)}$ if $a > 0, b > 0$.

(You may want to prove at the same time exercise (2), although it is possible to solve this question without appeal to the Fundamental theory of arithmetic.)

2. Let $a = p_1^{r_1} p_2^{r_2} \cdots p_k^{r_k}$ and $b = p_1^{s_1} p_2^{s_2} \cdots p_k^{s_k}$, where $p_1, p_2, \dots, p_k$ are distinct positive primes and each $r_i, s_i \geq 0$. Prove that
   (1) $(a, b) = p_1^{n_1} p_2^{n_2} \cdots p_k^{n_k}$, where $n_i = \min(r_i, s_i)$.
   (2) $[a, b] = p_1^{t_1} p_2^{t_2} \cdots p_k^{t_k}$, where $t_i = \max(r_i, s_i)$.

3. Prove or disprove: If n is an integer and $n > 2$, then there exists a prime p such that $n < p < n!$.

4. Find all the primes between 1 and 150. The solution should consist of a list of all the primes + giving the last prime used to sieve + explanation why you didn't have to sieve by larger primes.

5. Prove that
   (1) $\sqrt{2+\sqrt{3}}$ is irrational;
   (2) $\sqrt{2}+\sqrt{3}$ is irrational;
   (3) $\sqrt[3]{p}$ is irrational, if p is a prime.

6. A relation can be either reflexive or not, symmetric of not, transitive or not. This gives a priori 8 possibilities (e.g., reflexive, non-symmetric, transitive). For each possibility either give an example of such a relation, or indicate why this possibility doesn't occur.

7. Consider the relation on $\mathbb{Z}$ defined by $a \sim b$ if $a|b$. Is it reflexive? symmetric? transitive?

8. Calculate the following expressions, without using a calculator:
   (1) $1001 * 32 + 35 * 7921$ modulo 2.
   (2) $101 * 100 * 99 - 67$ modulo 102.
   (3) $7^{23} - 9^{24}$ modulo 8.
   (4) 2^{64} modulo 19. (Hint: calculate $2^2, 2^4, 2^8, 2^{16}, \dots$ by repeated squaring.)
   (5) 2^{66} modulo 19.