ASSIGNMENT 2 - MATH235, FALL 2009

Submit by 16:00, Monday, September 21 (use the designated mailbox in Burnside Hall, 10th floor).

1. Consider $\mathbb{N} \times \mathbb{N}$ as a rectangular array:

\[
\begin{array}{cccc}
(0, 0) & (0, 1) & (0, 2) & (0, 3) \\
(1, 0) & (1, 1) & (1, 2) & (1, 3) \\
(2, 0) & (2, 1) & (2, 2) & (2, 3) \\
(3, 0) & (3, 1) & (3, 2) & (3, 3) \\
\vdots
\end{array}
\]

Count the pairs in this array using diagonals as follows:

\[
\begin{array}{cccc}
0 & 1 & 3 & 6 & 10 \\
2 & 4 & 7 & 11 \\
5 & 8 & 12 \\
9 & 13 \\
14 \\
\vdots
\end{array}
\]

This defines a function

\[ f : \mathbb{N} \times \mathbb{N} \to \mathbb{N} \]

where $f(m, n)$ is the number appearing in the $(m, n)$ place. (For example, $f(0, 0) = 0$, $f(3, 1) = 13$, $f(2, 2) = 12$.) Provide an explicit formula for $f$ (it is what one calls “a polynomial function in the variables $m, n$”). It may be a good idea to first find a formula for $f(0, n)$.

2. Prove that if $|A_1| = |A_2|$ and $|B_1| = |B_2|$ then $|A_1 \times B_1| = |A_2 \times B_2|$.

3. Prove that $|\mathbb{N}| = |\mathbb{Q}|$. (Hint: Show two inequalities; note that there is an easy injection $\mathbb{Q} \to \mathbb{Z} \times \mathbb{Z}$).

4. Prove or disprove: if $|A_1| = |A_2|$ and $|B_1| = |B_2|$ then $|A_1 \setminus B_1| = |A_2 \setminus B_2|$.

5. Let $A$ be a set. Then $|A| < |2^A|$, where $2^A$ is the set of all subsets of $A$. Prove this as follows:

First show $|A| \leq |2^A|$ by constructing an injection $A \to 2^A$. Suppose now that there is a bijection $A \to 2^A$, $a \mapsto U_a$.

Define a subset $U$ of $A$ by

\[ U = \{ a : a \notin U_a \}. \]

Show that if $U = U_b$ we get a contradiction. (This is some sort of “diagonal argument”). Put all this together to conclude $|A| < |2^A|$.
Remark: If we take $A_1 = \mathbb{N}$, $A_2 = 2^\mathbb{N}$, and, inductively, $A_{n+1} = 2^{A_n}$ (the set whose elements are the subsets of $A_n$) then we find a series of increasing cardinalities. That shows that there is no set with maximal cardinality. Another way to argue that is: suppose there is a set $A$ with maximal possible cardinality. Then $|A| < |2^A|$ and $2^A$ is also a set. Contradiction.)

6. Prove that if a product $z_1 \cdot z_2$ of complex numbers is equal to zero then at least one of $z_1, z_2$ is zero.

7. Let $f$ be the complex polynomial $f(x) = x^2 + (1 + 6i)x + 1$ Find a complex number $z$ such that the equation $f(x) = z$ has a unique solution (use the formula for solving a quadratic equation).

8. Find the general form of a complex number $z$ such that $\Re(z)^2 = \Im(z)^2$ and $|z| \leq 1$. Also plot the answer on the complex plane.