ASSIGNMENT 2 - MATH235, FALL 2009

Submit by 16:00, Monday, September 21 (use the designated mailbox in Burnside Hall, 10^{th} **floor).**

1. Consider $\mathbb{N} \times \mathbb{N}$ as a rectangular array:

$$\begin{array}{ccccc} (0,0) & (0,1) & (0,2) & (0,3) & \dots \\ (1,0) & (1,1) & (1,2) & (1,3) & \dots \\ (2,0) & (2,1) & (2,2) & (2,3) & \dots \\ (3,0) & (3,1) & (3,2) & (3,3) & \dots \\ \vdots & & & & \end{array}$$

Count the pairs in this array using diagonals as follows:

$$\begin{array}{cccccc} 0 & 1 & 3 & 6 & 10 & \dots \\ 2 & 4 & 7 & 11 & & \dots \\ 5 & 8 & 12 & & & \dots \\ 9 & 13 & & & & \dots \\ 14 & & & & & \dots \\ \vdots & & & & & \end{array}$$

This defines a function

$$f : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$$

where $f(m,n)$ is the number appearing in the (m,n) place. (For example, $f(0,0) = 0, f(3,1) = 13, f(2,2) = 12$.) Provide an explicit formula for f (it is what one calls "a polynomial function in the variables m, n". It may be a good idea to first find a formula for $f(0,n)$).

2. Prove that if $|A_1| = |A_2|$ and $|B_1| = |B_2|$ then $|A_1 \times B_1| = |A_2 \times B_2|$.

3. Prove that $|\mathbb{N}| = |\mathbb{Q}|$. (Hint: Show two inequalities; note that there is an easy injection $\mathbb{Q} \to \mathbb{Z} \times \mathbb{Z}$).

4. Prove or disprove: if $|A_1| = |A_2|$ and $|B_1| = |B_2|$ then $|A_1 \setminus B_1| = |A_2 \setminus B_2|$.

5. Let A be a set. Then $|A| < |2^A|$, where 2^A is the set of all subsets of A. Prove this as follows: First show $|A| \le |2^A|$ by constructing an injection $A \to 2^A$. Suppose now that there is a bijection

$$A \to 2^A, \qquad a \mapsto U_a.$$

Define a subset U of A by

$$U = \{a : a \notin U_a\}.$$

Show that if $U = U_b$ we get a contradiction. (This is some sort of "diagonal argument"). Put all this together to conclude $|A| < |2^A|$.

(Remark: If we take $A_1 = \mathbb{N}$, $A_2 = 2^{\mathbb{N}}$, and, inductively, $A_{n+1} = 2^{A_n}$ (the set whose elements are the subsets of A_n) then we find a series of increasing cardinalities. That shows that there is no set with maximal cardinality. Another way to argue that is: suppose there is a set A with maximal possible cardinality. Then $|A| < |2^A|$ and 2^A is also a set. Contradiction.)

6. Prove that if a product $z_1 \cdot z_2$ of complex numbers is equal to zero then at least one of z_1, z_2 is zero.

7. Let f be the complex polynomial $f(x) = x^2 + (1+6i)x + 1$ Find a complex number z such that the equation $f(x) = z$ has a unique solution (use the formula for solving a quadratic equation).

8. Find the general form of a complex number z such that $\Re(z)^2 = \mathrm{Im}(z)^2$ and $|z| \leq 1$. Also plot the answer on the complex plane.