SOLUTIONS TO QUIZ 1 – ALGEBRA I, MATH235, FALL 2007

(1) Let A, B, C be sets. The set $(A \setminus (A \setminus B)) \cup (B \setminus A)$ is equal to B. (How to know? draw a diagram).

(2) The number of surjective functions from $\{1, 2, 3, 4\}$ to $\{1, 2\}$ is 14. (Why? There are altogether $16 = 2^4$ functions. Those that are not surjective take either only the value 1, or only the value 2. Thus there are 2 non surjective functions.)

(3) If $a|5c - d$ and $a|d - 3c$ and a is odd, then $a|c$ and $a|d$. (Why? a divides $(5c - d) + (d - 3c) = 2c$. Since a is odd $(a, 2) = 1$ and so $a|c$. Then $a|(d - 3c) + 3c$ so $a|d$).

(4) Let α be the cardinality of $\mathbb{R}$ and β the cardinality of the set of squares of natural numbers $\{0, 1, 4, 9, 16, 25, \dots, n^2, \dots\}$. Then $\alpha > \beta$. (The cardinality of the squares is the cardinality of $\mathbb{N}$. The inclusion $\mathbb{R} \supset \mathbb{N}$ gives $\alpha \geq \beta$ and we have proven in class (Cantor's diagonal argument) that $\alpha \neq \beta$.)

(5) One of the following lists consists entirely of irrational numbers. Which?
   (a) $\sqrt{2}, \sqrt{3}, \sqrt{4}, \sqrt{5}$.
   (b) $\sqrt{2}, \sqrt{\sqrt{2}}$.
   (c) $\sqrt{2}, (1 + i)^4$.
   (d) All the solutions to $x^3 = -1$ in complex numbers.
   The answer is $\{\sqrt{2}, \sqrt{\sqrt{2}}\}$. (Either by eliminating the rest: $\sqrt{4} = 2$, $(1 + i)^4 = -4$ ..., or by using that if $\sqrt{\sqrt{2}}$ is rational then so is its square, i.e., $\sqrt{2}$ is rational, but we know this is not so.)

(6) The gcd of 3990 and 546 is 42.

(7) Which of the following statements is correct?
   (a) The matrices $\left\{ \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} : a, d \in \mathbb{R}, b \in \mathbb{C} \right\}$ are a subring of $M_2(\mathbb{C})$.
   (b) $\{2n : n \in \mathbb{Z}\}$ is a subring of $\mathbb{Z}$.
   (c) $\{a + bi : a, b \in \mathbb{R}, ab = 0\}$ is a subring of $\mathbb{C}$.
   (d) The matrices $\left\{ \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} : a, d \in \mathbb{C}, b \in \mathbb{R} \right\}$ are a subring of $M_2(\mathbb{C})$.
   The correct answer is (a). In (b) we don't have 1. in (c) $1, i$ are elements of the set but not $1 + i$. In (d), the set is not closed under multiplication (typically, you'll get a complex entry for b after multiplication).

(8) Let A be the set of complex numbers z such that z^2 is real and B the set of complex numbers z such that $\bar{z} + z = 0$. Then $A \cap B = \{bi : b \in \mathbb{R}\}$. ($B$ is in fact the imaginary axis and is contained in A).

Answer the following question.

Prove that $\sqrt{2}$ is an irrational number.
Almost everyone followed one of the two proofs given in class. We insisted you be very precise (for example, when stating $\sqrt{2} = a/b$ saying also that $(a, b) = 1$ which is used later), or when writing $\sqrt{2} = p_1^{a_1} \cdots p_r^{a_r}$ that the p_i are distinct primes etc. On the whole we were very forgiving since this is the first test most of you take in a university. In the coming quizzes higher standards will be imposed).