ASSIGNMENT 9 - MATH235, FALL 2007

Submit by 16:00, Monday, November 19

(1) (a) Let Q_8 be the set of eight elements $\{\pm 1, \pm i, \pm j, \pm k\}$ in the quaternion ring $\mathbb{H}$, discussed in a previous assignment (so $ij = k = -ji$ etc.). Show that Q_8 is a group.
(b) For each of the groups S_3, D_4, Q_8 do the following:
(i) Write their multiplication table;
(ii) Find the order of each element of the group;
(iii) Find all the subgroups. Which of them are cyclic?

(2) (a) Find the order of the permutation $\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 3 & 1 & 5 & 6 & 2 & 7 & 8 & 4 \end{pmatrix}$ and write it as a product of cycles.
(b) Find a permutation in S_{12} of order 60. Is there a permutation of larger order in S_{12}?

(3) Let H_1, H_2 be subgroups of a group G. Prove that $H_1 \cap H_2$ is a subgroup of G.

(4) Let G be a group and let H_1, H_2 be subgroups of G. Prove that if $H_1 \cup H_2$ is a subgroup then either $H_1 \subseteq H_2$ or $H_2 \subseteq H_1$.

(5) Let $\mathbb{F}$ be a field. Prove that

$$\mathrm{SL}_2(\mathbb{F}) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d \in \mathbb{F}, ad - bc = 1 \right\}$$

is a group. If $\mathbb{F}$ has q elements, how many elements are in the group $\mathrm{SL}_2(\mathbb{F})$? (You may use the fact that $\mathrm{GL}_2(\mathbb{F})$ is a group, being the units of the ring $M_2(\mathbb{F})$.)

(6) Let $n \geq 2$ be an integer. Let S_n be the group of permutations on n elements $\{1, 2, ..., n\}$. A permutation σ is called a transposition if $\sigma = (i\ j)$ for some $i \neq j$, namely, σ exchanges i and j and leaves the rest of the elements in their places. Prove that every element of S_n is a product of transpositions.
Hint: Reduce to the case of cycles.

(7) (a) Let $n \geq 1$ integer. Show that the order of $a \in \mathbb{Z}/n\mathbb{Z}$, viewed as a group of order n with respect to addition, is $\frac{n}{\gcd(a,n)}$.
(b) Let $n \geq 3$. Find the order of every element of the dihedral group D_n.

(8) Let $n > 1$ be an integer, relatively prime to 10. Consider the decimal expansion of $1/n$. It is periodic, as we have proven in a previous assignment. Prove that the length of the period is precisely the order of 10 in the group $\mathbb{Z}/n\mathbb{Z}^{\times}$ (the group of congruence classes relatively prime to n, under multiplication).

For example: $1/3 = 0.33333\ldots$ has period 1 and the order of $10 \pmod 3 = 1 \pmod 3$, with respect to multiplication, is 1. We have $1/7 = 0.142857142857142857142857 1429\cdots$, which has period 6; the order of $10 \pmod 7 = 3 \pmod 7$ is 6 as we may check: $3, 3^2 = 9 \equiv 2, 3^3 = 6, 3^4 = 18 \equiv 4, 3^5 = 12 \equiv 5, 3^6 = 15 \equiv 1$.

Hint: how does one calculates the decimal expansion in practice?

(9) (a) Prove that $\mathbb{Z}_2 \times \mathbb{Z}_3 \cong \mathbb{Z}_6$.
(b) Prove that $\mathbb{Z}_6 \ncong S_3$, though both groups have 6 elements.
(c) Prove that the following groups of order 8 are not isomorphic: $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2, \mathbb{Z}_2 \times \mathbb{Z}_4, \mathbb{Z}_8, D_4, Q$. (Hint: use group properties such as "commutative" , "exists an element of order k", "number of elements of order k",...)

Remark: One can prove that every group of order 6 is isomorphic to $\mathbb{Z}_6$ or S_3. One can prove that every group of order 8 is isomorphic to one in our list.