ASSIGNMENT 8 - MATH235, FALL 2007

Submit by 16:00, Monday, November 12 (use the designated mailbox in Burnside Hall, 10^{th} **floor).**

1. Let $\mathbb{F}$ be a field and let

$$R = \left\{ \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{pmatrix} : a_{ij} \in \mathbb{F} \right\}.$$

Let

$$I = \{(a_{ij}) \in R : a_{11} = a_{22} = a_{33} = 0\}.$$

Prove that R is a subring of $M_3(\mathbb{F})$, I is an ideal of R and $R/I \cong \mathbb{F} \times \mathbb{F} \times \mathbb{F}$.

2. Prove a Chinese Remainder Theorem for polynomials:

Let $\mathbb{F}$ be a field and let $f(x), g(x)$ be two non-constant polynomials that are relatively prime, $\gcd(f, g) = 1$. Prove that

$$\mathbb{F}[x]/(fg) \cong \mathbb{F}[x]/(f) \times \mathbb{F}[x]/(g).$$

(Hint: mimic the proof of the Chinese Remainder Theorem of integers.)

3. Let R and S be rings and let $I \lhd R$, $J \lhd S$ be ideals. Prove that

$$(R \times S)/(I \times J) \cong (R/I) \times (S/J).$$

4. Using the Chinese remainder theorem, find the solutions (if any) to the following polynomial equations (for example, in (4), write the solutions as integers mod 30 and so on):

(1) $15x = 11$ in $\mathbb{Z}/18\mathbb{Z}$.
(2) $15x = 12$ in $\mathbb{Z}/63\mathbb{Z}$.
(3) $x^2 = 37$ in $\mathbb{Z}/63\mathbb{Z}$.
(4) $x^2 = 4$ in $\mathbb{Z}/30\mathbb{Z}$.

5. Let $f(x) = x^2 - 2 \in \mathbb{F}_{19}[x]$.

(1) Prove that f is irreducible and deduce that $L := \mathbb{F}_{19}[x]/(x^2 - 2)$ is a field.
(2) Find the roots of the polynomial $t^2 - t + 2$ in the field L.
(3) Prove that $h(x) = x^3 - 2$ is irreducible over $\mathbb{F}_{19}$. Prove that it is also irreducible in L.
(4) Find $x^{19} - x$ in $\mathbb{F}_{19}[x]/(x^3 - 2)$ as being represented by a polynomial of degree at most 2 (hint: $x^{19} = (x^3)^6 \cdot x$). Use this to rapidly calculate $\gcd(x^{19} - x, h(x))$ and conclude also in this way that $h(x)$ is irreducible over $\mathbb{F}_{19}$.

6. Write that following permutations as a product of disjoint cycles in S_9 and find their order:

(1) $\sigma\tau^2\sigma$, where $\sigma = (1234)(68)$ and $\tau = (123)(398)(45)$.
(2) $\sigma\tau\sigma\tau$, $\sigma = (123)$, $\tau = (345)(17)$.
(3) $\sigma^{-1}\tau\sigma$, $\sigma = (123456789)$, $\tau = (12)(345)(6789)$.
(4) $\sigma^{-1}\tau\sigma$, $\sigma = (123456789)$, $\tau = (12345)(6789)$.

7. Which of the following are subgroups of S_4?

(1) $\{1, (12)(34), (13)(24), (14)(23)\}$.
(2) $\{1, (1234), (13), (24), (13)(24)\}$.
(3) $\{1, (423), (432), (42), (43), (23)\}$.
(4) $\{1, (123), (231), (124), (142)\}$.