ASSIGNMENT 7 - MATH235, FALL 2007

Submit by 16:00, Monday, November 5

1. **The ring of real quaternions** $\mathbb{H}$**.** Let i, j, k be formal symbols and

$$\mathbb{H} = \{a + bi + cj + dk : a, b, c, d \in \mathbb{R}\}.$$

Addition on $\mathbb{H}$ is defined by

$$(a + bi + cj + dk) + (a' + b'i + c'j + d'k) = (a + a') + (b + b')i + (c + c')j + (d + d')k.$$

Multiplication is determined by defining

$$i^2 = j^2 = -1, \quad ij = -ji = k,$$

(and one extend this to a product rule by linearity).

(1) Prove that the map

$$\mathbb{H} \to \left\{ \begin{pmatrix} z_1 & z_2 \\ -\overline{z_2} & \overline{z_1} \end{pmatrix} : z_1, z_2 \in \mathbb{C} \right\},$$

taking $a + bi + cj + dk$ to the matrix $\begin{pmatrix} a + bi & c + di \\ -c + di & a - bi \end{pmatrix}$ is bijective and satisfies: $f(x + y) = f(x) + f(y)$ and $f(i)^2 = f(j)^2 = -I_2, f(i)f(j) = f(k) = -f(j)f(i)$.

(2) Use the previous question to conclude that $\mathbb{H}$ is a ring.

(3) Prove that $\mathbb{H}$ is a non-commutative division ring.

2.

(1) Prove that there is no ring homomorphism $\mathbb{Z}_5 \to \mathbb{Z}$.

(2) Prove that there is no ring homomorphism $\mathbb{Z}_5 \to \mathbb{Z}_7$.

3.

(1) Let R be any commutative ring and let $a_1, \dots, a_n$ be elements of R. We define $(a_1, \dots, a_n)$ to be the set

$$\{r_1 a_1 + \cdots + r_n a_n : \forall i \; r_i \in R\}.$$

Prove that $(a_1, \dots, a_n)$ is an ideal of R. We call it the ideal generated by $a_1, \dots, a_n$.

(2) Now apply that to the case where $R = \mathbb{Z}[x]$ (polynomials with integer coefficients). Let $(2, x)$ be the ideal generated by 2 and x.

(i) Prove that the ideal $(2, x)$ is not principal and conclude that $\mathbb{Z}[x]$ is not a principal ideal ring.

(ii) Find a homomorphism $f : R \to \mathbb{Z}_2$ such that $(2, x) = \mathrm{Ker}(f)$.

4. Prove that no two of the following rings are isomorphic:

(1) $\mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}$ (with addition and multiplication given coordinate by coordinate);

(2) $M_2(\mathbb{R})$;

(3) The ring $\mathbb{H}$ of real quaternions.

5. Let $f : R \to S$ be a ring homomorphism.

(1) Let $J \lhd S$ be an ideal. Prove that $f^{-1}(J)$ (equal by definition to $\{r \in R : f(r) \in J\}$) is an ideal of R.

(2) Prove that if f is surjective and $I \lhd R$ is an ideal then $f(I)$ is an ideal (where $f(I) = \{f(i) : i \in I\}$).

(3) Show, by example, that if f is not surjective the assertion in (2) need not hold.