ASSIGNMENT 6 - MATH235, FALL 2007

Submit by 16:00, Monday, October 29

1. Calculate the following:
   (1) $(2^{19808} + 6)^{-1} + 1 \pmod{11}$.
   (2) $12, 12^2, 12^4, 12^8, 12^{16}, 12^{25}$ all modulo 29. (Hint: think before computing).

2. Use the Euclidean algorithm to find the gcd of the following pairs of polynomials and express it as a combination of the two polynomials.
   (1) $x^4 - x^3 - x^2 + 1$ and $x^3 - 1$ in $\mathbb{Q}[x]$.
   (2) $x^5 + x^4 + 2x^3 - x^2 - x - 2$ and $x^4 + 2x^3 + 5x^2 + 4x + 4$ in $\mathbb{Q}[x]$.
   (3) $x^4 + 3x^3 + 2x + 4$ and $x^2 - 1$ in $\mathbb{Z}/5\mathbb{Z}[x]$.
   (4) $4x^4 + 2x^3 + 3x^2 + 4x + 5$ and $3x^3 + 5x^2 + 6x$ in $\mathbb{Z}/7\mathbb{Z}[x]$.
   (5) $x^3 - ix^2 + 4x - 4i$ and $x^2 + 1$ in $\mathbb{C}[x]$.
   (6) $x^4 + x + 1$ and $x^2 + x + 1$ in $\mathbb{Z}/2\mathbb{Z}[x]$.

3. Consider the polynomial $x^2 + x = 0$ over $\mathbb{Z}/n\mathbb{Z}$.
   (1) Find an n such that the equation has at least 4 solutions.
   (2) Find an n such that the equation has at least 8 solutions.

4. Is the given polynomial irreducible:
   (1) $x^2 - 3$ in $\mathbb{Q}[x]$? In $\mathbb{R}[x]$?
   (2) $x^2 + x - 2$ in $\mathbb{F}_3[x]$? In $\mathbb{F}_7[x]$?

5. Find the rational roots of the polynomial $2x^4 + 4x^3 - 5x^2 - 5x + 2$.

6. Recall that for the ring $\mathbb{Z}$ a complete list of ideals is given by $(0), (1), (2), (3), (4), (5), \dots$, where (n) is the principal ideal generated by n, namely, $(n) = \{na : a \in \mathbb{Z}\}$. Find the complete list of ideals of the ring $\mathbb{Z} \times \mathbb{Z}$.

7. Let R be a ring and let I and J be two ideals of R.
   (1) Prove that $I \cap J$ is an ideal of R, where
   $$I \cap J = \{r : r \in I, r \in J\}$$
   (the intersection of the sets). It is called the intersection of the ideals I and J.
   (2) Prove that
   $$I + J = \{i + j : i \in I, j \in J\}$$
   is an ideal of R. It is called the sum of the ideals I and J.
   (3) Find for every two ideals of the ring $\mathbb{Z}$ their sum and intersection.

8. Let $\mathbb{F}$ be a field. Prove that the ring $M_2(\mathbb{F})$ of 2×2 matrices with entries in $\mathbb{F}$ has no non-trivial (two-sided) ideals. That is, every ideal is either the zero ideal or $M_2(\mathbb{F})$ itself.
(Note: there is also a notion of a one-sided ideal that we don't discuss in this course. The ring $M_2(\mathbb{F})$ has a non-trivial one sided ideal. The notion of one-sided ideals is usually studied in MATH570, MATH571).