1. To check if you had multiplied correctly two large numbers $A$ and $B$, $A \times B = C$, you can make the following check: sum the digits of $A$; keep doing it repeatedly until you get a single digit number $a$. Do the same for $B$ and $C$ and get numbers $b, c$. If you have multiplied correctly, the sum of digits of $ab$ is $c$. Prove that this is so. This is called in French “preuve par neuf”.

Example: I have multiplied $A = 367542$ by $B = 687653$ and got $C = 252741358926$. To check (though this doesn’t prove the multiplication is correct) I do: $3 + 6 + 7 + 5 + 4 + 2 = 27$, $2 + 7 = 9$ and $a = 9$. Also $6 + 8 + 7 + 6 + 5 + 3 = 35$, $3 + 5 = 8$ and $b = 8$. $ab = 72$ and its sum of digits is 9. On the other hand $2 + 5 + 2 + 7 + 4 + 1 + 3 + 5 + 8 + 9 + 2 + 6 = 54$, $5 + 4 = 9$. So it checks.

2. (1) Solve that equation $x^2 + x = 0$ in $\mathbb{Z}/5\mathbb{Z}$.
   (2) Solve that equation $x^2 + x = 0$ in $\mathbb{Z}/6\mathbb{Z}$.
   (3) Solve that equation $x^2 + x = 0$ in $\mathbb{Z}/p\mathbb{Z}$, where $p$ is prime.

3. Solve each of the following equations:
   (1) $12x = 2$ in $\mathbb{Z}/19\mathbb{Z}$.
   (2) $7x = 2$ in $\mathbb{Z}/24\mathbb{Z}$.
   (3) $31x = 1$ in $\mathbb{Z}/50\mathbb{Z}$.
   (4) $34x = 1$ in $\mathbb{Z}/97\mathbb{Z}$.
   (5) $27x = 2$ in $\mathbb{Z}/40\mathbb{Z}$.
   (6) $15x = 5$ in $\mathbb{Z}/63\mathbb{Z}$.

4. (1) Let $p > 2$ be a prime. Prove that an equation of the form $ax^2 + bx + c$ (where $a, b, c \in \mathbb{F}_p, a \neq 0$) has a solution in $\mathbb{Z}/p\mathbb{Z}$ if and only if $b^2 - 4ac$ is a square in $\mathbb{Z}/p\mathbb{Z}$. If this is so, prove that the solutions are given by the familiar formula.
   (2) Determine for which values of $a$ the equation $x^2 + x + a$ has a solution in $\mathbb{Z}/7\mathbb{Z}$.

5. In each case, divide $f(x)$ by $g(x)$ with residue:
   (1) $f(x) = 3x^4 - 2x^3 + 6x^2 - x + 2$, $g(x) = x^2 + x + 1$ in $\mathbb{Q}[x]$.
   (2) $f(x) = x^4 - 7x + 1$, $g(x) = 2x^2 + 1$ in $\mathbb{Q}[x]$.
   (3) $f(x) = 2x^4 + x^3 - x + 1$, $g(x) = 2x - 1$ in $\mathbb{Z}/5\mathbb{Z}[x]$.
   (4) $f(x) = 4x^4 + 2x^3 + 6x^2 + 4x + 5$, $g(x) = 3x^2 + 2$ in $\mathbb{Z}/7\mathbb{Z}[x]$. 