ASSIGNMENT 4 - MATH235, FALL 2007

Submit by 16:00, Tuesday, October 9 (use the designated mailbox in Burnside Hall, 10^{th} floor).

1. Let $a = p_1^{r_1} p_2^{r_2} \cdots p_k^{r_k}$ and $b = p_1^{s_1} p_2^{s_2} \cdots p_k^{s_k}$, where $p_1, p_2, \ldots, p_k$ are distinct positive primes and each $r_i, s_i \geq 0$. Using unique factorization, prove that
   (1) $(a, b) = p_1^{n_1} p_2^{n_2} \cdots p_k^{n_k}$, where $n_i = \min(r_i, s_i)$.
   (2) $[a, b] = p_1^{t_1} p_2^{t_2} \cdots p_k^{t_k}$, where $t_i = \max(r_i, s_i)$.

2. The least common multiple of nonzero integers a, b is the smallest positive integer m such that $a|m$ and $b|m$. We denote it by $\operatorname{lcm}(a, b)$ or $[a, b]$. Prove that:
   (1) If $a|k$ and $b|k$ then $[a, b]|k$.
   (2) $[a, b] = \frac{ab}{(a,b)}$ if $a > 0, b > 0$.

3. Prove or disprove: If n is an integer and $n > 2$, then there exists a prime p such that $n < p < n!$.

4. Find all the primes between 1 and 150. The solution should consist of a list of all the primes + giving the last prime used to sieve + explanation why you didn't have to sieve by larger primes.

5. Prove that $\sqrt{2 + \sqrt{3}}$ is irrational.

6. Given an integer N we write N in decimal expansion as $N = n_k n_{k-1} \ldots n_0$, the n_i being the digits of N. Note that this means that $N = n_0 + 10n_1 + 10^2 n_2 + \cdots + 10^k n_k$. In the following you are asked to show certain divisibility criteria that can be proved by using congruences.
   (1) Prove that a positive integer $N = n_k n_{k-1} \ldots n_0$ is divisible by 3 if and only if the sum of its digits $n_0 + n_1 + \cdots + n_k$ is divisible by 3. (Hint: show that in fact N and $n_0 + n_1 + \cdots + n_k$ are congruent to the same number modulo 3.) Example: 34515 is divisible by 3 because $3 + 4 + 5 + 1 + 5 = 18$ is divisible by 3.
   (2) Prove that a positive integer $N = n_k n_{k-1} \ldots n_0$ is divisible by 11 if and only if the sum of its digits with alternating signs $n_0 - n_1 + n_2 - \cdots \pm n_k$ is divisible by 11. The same Hint applies here. Example: 1234563 is divisible by 11 since $1 - 2 + 3 - 4 + 5 - 6 + 3 = 0$ is divisible by 11.
   (3) Prove that a positive integer $N = n_k n_{k-1} \ldots n_0$ is divisible by 7 if and only if when we let $M = n_k n_{k-1} \ldots n_1$, we have $M - 2n_0$ is divisible by 7. Example: take the number $7 * 11 * 13 * 17 = 17017$. It is clearly divisible by 7. Let us check the criterion against this example. We form the number $1701 - 2 * 7 = 1687$ and then the number $168 - 2 * 7 = 154$ and then the number $15 - 2 * 4 = 7$. So it works. Let us also check the number 82. It is not divisible by 7, in fact it's residue modulo 7 is 5. Also $8 - 2 * 2 = 4$, so the criterion shows that it's not divisible. Note though that in this case the number $N = 82$ and the number $M = 8 - 2 * 2 = 4$ don't have the same residue modulo 7. So you need to construct your argument a little differently.

7. Prove that if p is a prime then $\sqrt{p}$ is irrational.