ASSIGNMENT 3 - MATH235, FALL 2007

Submit by 16:00, Monday, October 1 (use the designated mailbox in Burnside Hall, 10th floor).

1. The ring of $2 \times 2$ matrices over a field $F$.

Let $F$ be a field. (In a first read you may as well assume $F$ is $\mathbb{R}$, but solve the question in general). We consider the set

$$ M_2(F) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d \in F \right\}. $$

It is called the two-by-two matrices over $F$. We define the addition of two matrices as

$$ \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix} + \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix} = \begin{pmatrix} a_1 + a_2 & b_1 + b_2 \\ c_1 + c_2 & d_1 + d_2 \end{pmatrix}. $$

We define multiplication by

$$ \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix} \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix} = \begin{pmatrix} a_1a_2 + b_1c_2 & a_1b_2 + b_1d_2 \\ c_1a_2 + d_1c_2 & c_1b_2 + d_1d_2 \end{pmatrix}. $$

Prove that this is a ring. For each of the following subsets of $M_2(F)$ determine if they are subrings or not.

(A) The set $\left\{ \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \in M_2(F) \right\}$.

(B) The set $\left\{ \begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix} \in M_2(F) \right\}$.

(C) The set $\left\{ \begin{pmatrix} a & 0 \\ c & d \end{pmatrix} \in M_2(F) \right\}$.

(D) The set $\left\{ \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} \in M_2(F) \right\}$.

(E) The set $\left\{ \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} \in M_2(F) \right\}$.

Remark: One defines in a very similar way the ring of $n \times n$ matrices with entries in a field $F$.

2. Find the quotient and remainder when $a$ is divided by $b$:

(1) $a = 302, b = 19$.

(2) $a = -302, b = 19$.

(3) $a = 0, b = 19$.

(4) $a = 2000, b = 17$.

(5) $a = 2001, b = 17$.

(6) $a = 2002, b = 17$.

3. Prove that the square of any integer $a$ is either of the form $3k$ or of the form $3k + 1$ for some integer $k$. (Hint: write $a$ in the form $3q + r$, where $r = 0, 1$ or $2$.)

4. Prove of disprove: If $a|(b + c)$ then $a|b$ or $a|c$.

5. If $r \in \mathbb{Z}$ and $r$ is a nonzero solution of $x^2 + ax + b$ (where $a, b \in \mathbb{Z}$) prove that $r|b$.

6. If $n \in \mathbb{Z}$, what are the possible values of

(1) $(n, n + 2)$;

(2) $(n, n + 6)$.

7. Find the following gcd’s. In each case also express $(a, b)$ as $ua + vb$ for suitable integers $u, v \in \mathbb{Z}$.

(1) $(56, 72)$.

(2) $(24, 138)$.

(3) $(143, 227)$.

(4) $(314, 159)$.

8. If $a|c$ and $b|c$, must $ab$ divide $c$? What if $(a, b) = 1$?