(1) Let $G$ be a finite group acting on a finite set. Prove that if $\langle a \rangle = \langle b \rangle$ for two elements $a, b \in G$ then $I(a) = I(b)$, where for any element $g \in G$, $I(g) = |\{s \in S : gs = s\}|$.

(2) Let $p$ be a prime number. Let $G$ be a finite group of $p^r$ elements. Let $S$ be a finite set having $N$ elements and assume that $(p, N) = 1$. Assume that $G$ acts on $S$. Prove that $G$ has a fixed point in $S$. Namely, there exists $s \in S$ such that $g \star s = s$ for every $g \in G$.

(3) Find how many necklaces with 4 Rubies, 5 Sapphires and 3 Diamonds are there, up to the usual identifications.

(4) Let $G = S_3$ be the group of permutations of three elements, and define an action of $G$ on the set $S = \mathbb{Z}/5\mathbb{Z} \times \mathbb{Z}/5\mathbb{Z} \times \mathbb{Z}/5\mathbb{Z}$ by

$\sigma \star (x_1, x_2, x_3) = (x_{\sigma^{-1}(1)}, x_{\sigma^{-1}(2)}, x_{\sigma^{-1}(3)})$.

For example, if $\sigma = (132)$ then $\sigma^{-1} = (123)$ and $\sigma \star (1, 4, 5) = (4, 5, 1)$; if $\sigma = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}$ then $\sigma \star (1, 4, 5) = (4, 1, 5)$. Calculate the number of orbits for this action.

(5) Find all the homomorphic images of the groups $D_4, Q$. (Guidance: If $G$ is one of these groups, show that this amounts to classifying the normal subgroups of $G$ and for each such normal subgroup $H$ find an isomorphism of $G/H$ with a group known to us.)

(6) Let $R$ be the ring of matrices

$\left\{ \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{pmatrix} : a_{ij} \in \mathbb{Z}_2 \right\}$.

Note that $R$ has $2^6 = 64$ elements. Let $G$ be the group of units of $R$. In this case it consists of the matrices such that $a_{11} = a_{22} = a_{33} = 1$. Therefore $G$ has 8 elements. Prove that $G \cong D_4$.

(7) Let $H$ be a subgroup of index 2 of a group $G$. Prove that $H$ is normal in $G$. 

Submit by 16:00, Monday, December 3