ASSIGNMENT 10 - MATH235, FALL 2007

Submit by 16:00, Monday, December 3

(1) Let G be a finite group acting on a finite set. Prove that if $\langle a \rangle = \langle b \rangle$ for two elements $a, b \in G$ then $I(a) = I(b)$, where for any element $g \in G$, $I(g) = |\{s \in S : gs = s\}|$.

(2) Let p be a prime number. Let G be a finite group of p^r elements. Let S be a finite set having N elements and assume that $(p, N) = 1$. Assume that G acts of S. Prove that G has a fixed point in S. Namely, there exists $s \in S$ such that $g \star s = s$ for every $g \in G$.

(3) Find how many necklaces with 4 Rubies, 5 Sapphires and 3 Diamonds are there, up to the usual identifications.

(4) Let $G = S_3$ be the group of permutations of three elements, and define an action of G on the set $S = \mathbb{Z}/5\mathbb{Z} \times \mathbb{Z}/5\mathbb{Z} \times \mathbb{Z}/5\mathbb{Z}$ by

$$\sigma \star (x_1, x_2, x_3) = (x_{\sigma^{-1}(1)}, x_{\sigma^{-1}(2)}, x_{\sigma^{-1}(3)}).$$

For example, if $\sigma = (132)$ then $\sigma^{-1} = (123)$ and $\sigma \star (1, 4, 5) = (4, 5, 1)$; if $\sigma = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}$ then $\sigma \star (1, 4, 5) = (4, 1, 5)$. Calculate the number of orbits for this action.

(5) Find all the homomorphic images of the groups D_4, Q. (Guidance: If G is one of these groups, show that this amounts to classifying the normal subgroups of G and for each such normal subgroup H find an isomorphism of G/H with a group known to us.)

(6) Let R be the ring of matrices

$$\left\{ \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{pmatrix} : a_{ij} \in \mathbb{Z}_2 \right\}.$$

Note that R has $2^6 = 64$ elements. Let G be the group of units of R. In this case it consists of the matrices such that $a_{11} = a_{22} = a_{33} = 1$. Therefore G has 8 elements. Prove that $G \cong D_4$.

(7) Let H be a subgroup of index 2 of a group G. Prove that H is normal in G.