PART A. (50 points for answering correctly 5 out of 6).

(1) Which of the following functions is a ring homomorphism from \( \mathbb{R}[x] \to \mathbb{R} \)? (circle all correct answers)

(a) The map taking \( f(x) = a_n x^n + \cdots + a_1 x + a_0 \) to \( a_0 \).
(b) The map taking \( f(x) = a_n x^n + \cdots + a_1 x + a_0 \) to \( a_1 + a_0 \).
(c) The map taking \( f(x) = a_n x^n + \cdots + a_1 x + a_0, a_n \neq 0 \), to \( a_n \) and taking the zero polynomial to zero.
(d) The map taking \( f(x) = a_n x^n + \cdots a_1 x + a_0 \) to \( a_0 - a_1 + a_2 - \cdots + (-1)^n a_n \).

The correct answers are (a) and (d). They are special cases of evaluation homomorphisms. The first is by substituting 0 for \( x \), the second is by substituting \(-1\) for \( x \).

(2) Circle all correct statements:

(a) Every ideal of \( \mathbb{Z} \) is principal.
(b) Every ideal of \( \mathbb{Z}[x] \) is principal.
(c) Every ideal of \( \mathbb{Q} \) is principal.
(d) Every ideal of \( \mathbb{Q}[x] \) is principal.
(e) Every ideal of \( \mathbb{Q}[x, y] \) is principal.
(f) Every ideal of \( \mathbb{Q}[x]/(x^2 + x + 1) \) is principal.
(g) Every ideal of \( \mathbb{Q}[x]/(x^2) \) is principal.

The correct answers are (a), (c), (d), (f), (g). For (a) and (d) these are theorems we proved. In case (c) and (f) we deal with fields \( x^2 + x + 1 \) is irreducible over \( \mathbb{Q} \) and a field \( \mathbb{F} \) has only two ideals \((0)\) and \((1) = \mathbb{F} \) (because any non-zero ideal contains a unit and so is equal to \( \mathbb{F} \)). Case (g) (which is just the ring \( \mathbb{Q}[\epsilon] \)) was analyzed in class. The cases (b) and (e) were discussed in the notes, tutorials and/or assignments.
(3) Let $\sigma = (1234)$, $\tau = (15)$ be permutations in the symmetric group $S_5$. Then $\sigma^2\tau\sigma^{-1}$ is

(a) $(12453)$.
(b) $(12)(45)$.
(c) $(12534)$.
(d) $(134)(25)$.

The correct answer is $(12534)$.

(4) The maximal order an element of $S_7$ can have is

(a) 7.
(b) 12.
(c) 14.
(d) 8.
(e) 10.

The answer is 12, realized for example by $(1234)(567)$.

(5) Which is of the following is the correct statement? (only one statement is correct)

(a) The ring $\mathbb{R}[x]/(x^2 + 1)$ is isomorphic to $\mathbb{C}$, the isomorphism taking $x$ to $i$.
(b) The ring $\mathbb{R}[x]/(x^2 + 1)$ is isomorphic to $\mathbb{R} \times \mathbb{R}$, the isomorphism taking $x$ to $(0,1)$.
(c) The ring $\mathbb{R}[x]/(x^2 + 1)$ is isomorphic to $\mathbb{R}$, the isomorphism taking $x$ to 0.
(d) The ring $\mathbb{R}[x]/(x^2 + 1)$ is isomorphic to the ring of dual numbers $\mathbb{R}[\epsilon]$, the isomorphism taking $x$ to $\epsilon$.

The correct statement is (a). We did it in class and it’s in the notes too.

(6) The number of solutions for the equation $x^2 + x - 4 \equiv 0 \pmod{17 \cdot 19}$ is

(a) 1.
(b) 0.
(c) 2.
(d) 4.

The answer is 2. The discriminant is 17 and so there is one solution modulo 17. On the other hand $17 \equiv 36 \pmod{19}$ and so there are two solutions modulo 19. The number of solutions $\pmod{17 \cdot 19}$ is $1 \times 2 = 2$. 

PART B. (50 points) Answer the following question in the exam notebook. Make sure you quote explicitly and precisely the results you are using in your proof.

Let $\mathbb{F}_5 = \mathbb{Z}/5\mathbb{Z}$ be the field of 5 elements. Prove the following:

1. The polynomial $x^2 - 2$ is irreducible in $\mathbb{F}_5$.
2. The ring $L = \mathbb{F}_5[x]/(x^2 - 2)$ is a field of 25 elements.
3. The polynomial $t^2 - 2$ has a root in $L$. What is it?
4. Prove that the polynomial $f(t) = t^2 + t - 3$ is also irreducible over $\mathbb{F}_5$. Find a root of $f$ in $L$ (write it as $ax + b$ for some $a, b \in \mathbb{F}_5$).

The polynomial $x^2 - 2$ is irreducible in $\mathbb{F}_5$ because 2 is not a square in $\mathbb{F}_5$. In fact, the squares are 0, 1, 4.

We proved that if $\mathbb{F}$ is a field and $h(x)$ is an irreducible (non-constant) polynomial then $\mathbb{F}[x]/(h(x))$ is a field and so in particular $L$ is a field. We further proved that if $h$ has degree $n$, then any element of $\mathbb{F}[x]/(h(x))$ is represented by a unique polynomial class of degree less than $n$. In particular, if $\mathbb{F}$ has $q$ elements then $\mathbb{F}[x]/(h(x))$ has $q^n$ elements. Applying it to our situation we find that $L$ has $5^2 = 25$ elements.

A root of the polynomial $t^2 - 2$ in $L$ is $x$. Indeed, $x^2 - 2$ is zero in $L$. (Another root is $-x$).

The discriminant of $t^2 + t - 3$ is $1 + 4 \cdot 3 = 13 \equiv 3 \pmod{5}$, which is not a square in $\mathbb{F}_5$ (see above). Since a quadratic polynomial is reducible if and only if it has a root, and the roots of $f(t)$ are $2^{-1}(-1 \pm \sqrt{3})$, if $f(t)$ is reducible then 3 is a square in $\mathbb{F}_5$ and we know that’s not the case.

To show 3 is a square in $L$, we solve the equation $(ax + b)^2 = 3$ for $a, b \in \mathbb{F}_5$. One finds that $a^2x^2 + 2abx + b^2 = 2abx + 2a^2 + b^2 = 3$. It follows that $2ab = 0$ and so that either $a = 0$ or $b = 0$. If $a = 0$ it follows that $b^2 = 3$, which does not have a solution. Thus, $b = 0$ and we find that $2a^2 = 3$ and we can take $a = 2$. Thus $3 = (2x)^2$ in $L$. The solutions for $f(t)$ are then $2^{-1}(-1 \pm \sqrt{3}) = 3(-1 \pm 2x) = 2 \pm x$. 