Algebra 4 (2003-04) – Assignment 9

Instructor: Dr. Eyal Goren

Submit by Monday, March 22, 12:00.

1) Let p and ℓ be primes (equal or distinct). Let K be the splitting field over $\mathbb{Q}$ of the polynomial $x^p - \ell$. Prove that $K/\mathbb{Q}$ is Galois and that

$$\mathrm{Gal}(K/\mathbb{Q}) \cong \mathbb{Z}/p\mathbb{Z} \rtimes (\mathbb{Z}/p\mathbb{Z})^{\times}.$$

Guidance: First establish that $K/\mathbb{Q}$ is Galois using a theorem we proved (not the definition). Then show $[K:\mathbb{Q}] = p(p-1)$ and that $K = \mathbb{Q}(\sqrt[p]{\ell}, \zeta_p)$. Examine the possible action of automorphisms of K on $\sqrt[p]{\ell}, \zeta_p$, and show there are at most $p(p-1)$ possibilities. Conclude that they all occur. Using again the action on $\sqrt[p]{\ell}, \zeta_p$, calculate the commutation rules.

2) Let k be a field. Show that S_n acts as automorphisms of $k(t_1, \dots, t_n)$, the quotient field of $k[t_1, \dots, t_n]$. Use that to prove that every finite group G is the Galois group of some Galois extension of fields $K \supset L$.

Hint: what would you do for $G = S_n$?