Algebra 4 (2003-04) – Assignment 5

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Submit by Monday, February 16, 12:00 by mail-box on 10^{th} **floor.**

1) Prove that if $[F(\alpha):F]$ is odd then $F(\alpha) = F(\alpha^2)$. Can you generalize this statement?

2)[1]

- Let $K \supset F$ be an extension of fields of degree $[K:F] = n$. Choose a basis $v_1, \dots, v_n$ for K as a vector space over F. Given any $\alpha \in K$ we consider the map

  $$T(\alpha) : K \longrightarrow K, \qquad k \mapsto \alpha k;$$

  Verify that $T(\alpha)$ is an F-linear map. Thus, we can associate to each α an $n \times n$ matrix, namely, the matrix $M(\alpha)$ that represent $T(\alpha)$ with respect to the basis we have chosen. Prove that this gives an injective ring homomorphism $K \hookrightarrow M_n(F)$. We conclude that we can realize every extension of F of degree n as a subfield of the ring of matrices $M_n(F)$.
- Prove that α is a root of the characteristic polynomial $\Delta(M(\alpha))$ of $M(\alpha)$, in fact that the minimal polynomial $m(\alpha)$ of α divides $\Delta(M(\alpha))$.
- Use this method to calculate the minimal polynomial of $\sqrt[3]{2}$ and $1 + \sqrt[3]{2} + \sqrt[3]{4}$ over $\mathbb{Q}$.

[1]Cf. Dummit and Foote p. 531. Here is an example: Suppose $K = \mathbb{Q}(\sqrt{2})$ and choose the basis $v_1 = 1, v_2 = \sqrt{2}$. Then multiplication by $\alpha = \sqrt{2}$ takes v_1 to v_2 and v_2 to $2v_1$. Thus, $M(\alpha) = \begin{smallmatrix} 0 & 2 \\ 1 & 0 \end{smallmatrix}$, which has characteristic polynomial $x^2 - 2$. Multiplication by $\alpha = 1 + 5\sqrt{2}$ is given by the matrix $\begin{smallmatrix} 1 & 10 \\ 5 & 1 \end{smallmatrix}$, which has characteristic polynomial $x^2 - 2x - 49$ etc.