Algebra 4 (2003-04) – Assignment 2

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Submit by Monday, January 26, 12:00 by mail-box on 10^{th} floor.

Let R be a commutative ring with 1. An ideal $I \lhd R$ is called *nilpotent* if there is a positive integer k such that $I^k = 0$. Recalling the definition of a product of ideals, we see that I is nilpotent with $I^k = 0$ if and only if for any k elements $x_1, \dots, x_k$ of I we have $x_1 x_2 \cdots x_k = 0$.

Let R' be any commutative ring and J an ideal of R'. Let $R = R'/J^k$ and let I be the ideal which is the image of J under $R' \to R'/J^k$. Then I is nilpotent and $I^k = 0$. As a concrete example, the ideal generated by x in $\mathbb{F}[x]/(x^k)$ is nilpotent (of degree k).

1) Let I be a nilpotent ideal of R. Let M, N be R-modules and $f : M \longrightarrow N$ an R-module homomorphism. Prove that there is a well defined induced homomorphism of R modules $\bar{f} : M/IM \longrightarrow N/IN$. Prove that if $\bar{f}$ is surjective then f is surjective.

2) Assume that R is an integral domain and let

$$0 \longrightarrow M_1 \longrightarrow M_2 \longrightarrow \cdots \longrightarrow M_n \longrightarrow 0$$

be an exact sequence of R-modules. Prove that

$$\sum_i (-1)^i \mathrm{rank}_R(M_i) = 0.$$

<u>Guidance</u>: This requires some work. You may use Exercise 2 in Dummit and Foote, §12.1 p. 468. I suggest that you write yourself a proof of that exercise but you need not submit it. For how to use this exercise for our problem you may want to compare with loc. cit. Exercises 3, 4. (Note that both are special cases of what you need to prove, viz. $0 \to A \to A \oplus B \to B \to 0$, $\quad 0 \to N \to M \to M/N \to 0$.)