Algebra 4 (2003-04) – Assignment 1

Instructor: Dr. Eyal Goren

Submit by Monday, January 19, 12:00 by mail-box on 10^{th} floor.

1) Let R be a ring and let

$$0 \longrightarrow M_1 \longrightarrow M_2 \longrightarrow \cdots \longrightarrow M_n \longrightarrow 0$$

be an exact sequence of R-modules. Prove the following:

(1) If $R = \mathbb{Z}$ and the M_i are finite abelian groups then $\prod_i |M_i|^{(-1)^i} = 1$.

(2) If $R = \mathbb{F}$ is a field and the M_i are finite dimensional vector spaces then $\sum_i (-1)^i \dim_{\mathbb{F}}(M_i) = 0$.

2) Let R be a commutative ring. Prove that if $R^m \cong R^n$ as R-modules then $m = n$. Suggestion: use the fact that R always has a maximal ideal I and try to reduce to the case of modules over R/I. You may use results about vector spaces. [1]

$*$) Suggested Exercise (do NOT submit): Prove the Chinese Remainder Theorem for modules.

[1] Dummit and Foote, Exercise 27, p. 358 shows that the statement can be wrong for R not commutative.