Midterm Exam – Algebra 4 (MATH 371)

March 3, 2005

Time: 16:00 - 18:00.
Instructions: Answer all quetions in clear handwriting. Use of any material such as books, notebooks, calculators, is not allowed. The maximal possible grade is 110. The difficulty of a question is not necessarily proportional to the number of points it's worth!

1. Let R be an integral domain.

(1) (5 points) Define the *rank* of a finitely generated R module.
(2) (15 points) Let n be a positive integer. Prove that the rank of R^n is n.

2. (30 points) Let $K \supseteq F$ be an extension of fields. Let

$$H = \{\alpha \in K : \alpha \text{ is algebraic over } F\}.$$

Prove that H is a field and any element of K which is not in H is transcendental over H. (Here you may of course use a certain characterization of algebraic elements we proved in class.)

3. Determine the degree of the following splitting fields.

- (20 points) Prove that the splitting field of $x^4 - 3$ has degree 8 over $\mathbb{Q}$.
- (20 points) Find the degree of the splitting field of $x^4 - 4$ over $\mathbb{Q}$.
- (20 points) Prove that the degree of the splitting field of $x^4 - 4$ over $\mathbb{Z}/5\mathbb{Z}$ is 2.