Algebra 4 (2004-05) – Assignment 9

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Submit by Wednesday, March 30, 24:00 by mail-box on 10^{th} floor.

1. For each of the Galois groups $\mathrm{Gal}(K/F)$ you calculated in Assignment 7, Question 1, (1), (2), (3), (5), (6), find all the subfields $K \supseteq E \supseteq F$ under the Galois correspondence. Write each field E in the form $F(\alpha)$ and find the minimal polynomial of α.[1]

2. Assuming the existence of a Galois extension $K/\mathbb{Q}$ with Galois group $\mathrm{Gal}(K/\mathbb{Q}) = A_4$, prove that there is an extension $F/\mathbb{Q}$ such that $[F : \mathbb{Q}] = 4$ and there is no proper subfield $F \supset E \supset \mathbb{Q}$. Conclude that the necessary condition $[\mathbb{Q}(\ell) : \mathbb{Q}] = 2^n$ for constructibility of a length ℓ, is not a sufficient condition.

[1]The Galois groups are :(i) S_3; (ii) $\mathbb{Z}/2\mathbb{Z}$ ($x^3 - 1$ and $x^2 + 3$ has the same splitting field!); (iii) $(\mathbb{Z}/2\mathbb{Z})^2$; (iv) $\mathrm{PGL}_2(\mathbb{C})$ (the center acts trivially. Any automorphism has this form because we know the degree $[\mathbb{C}(t) : \mathbb{C}(f(t)/g(t))]$ is the maximum of the degrees of f and g (f/g assumes to be in reduced terms)); (v) $\mathbb{Z}/4\mathbb{Z}$; (vi) $\mathbb{Z}/2\mathbb{Z}$.