1) Let $f: \mathbb{Z}^n \rightarrow \mathbb{Z}^n$ be a group homomorphism represented with respect to the standard basis by a matrix $M \in M_n(\mathbb{Z})$. Assume that $\det(M) \neq 0$. Prove that $\#(\mathbb{Z}^n/f(\mathbb{Z}^n)) = |\det(M)|$.

2) Let $F$ be a field and $A \in M_n(F)$. We think of $A$ as defining an $F[x]$ module structure on $F^n$. Note that the structure theorem says that there is a basis for $F^n$ in which multiplication by $x$ is given by $\text{diag}(C_{a_1(x)}, C_{a_2(x)}, \ldots, C_{a_m(x)})$, where $a_1(x)|a_2(x)| \ldots |a_m(x)$ are the invariant factors and the $C_{a_i(x)}$ the companion matrices. That means that if we let $N$ be the change of basis matrix then $NAN^{-1} = \text{diag}(C_{a_1(x)}, C_{a_2(x)}, \ldots, C_{a_m(x)})$. Note that if for some other basis with change of basis matrix $N'$ we have that $N'AN'^{-1} = \text{diag}(C_{a_1'(x)}, C_{a_2'(x)}, \ldots, C_{a_{m'}(x)})$ with $a_1'(x)|a_2'(x)| \ldots |a_{m'}(x)$ then this gives a decomposition of the $F[x]$ module $F^n$ into a direct sum of modules of the form $F[x]/(a_i'(x))$ and thus (if all polynomials considered are monic) we have $m = m'$ and $a_i(x) = a_i'(x)$ for all $i$. Namely, $A$ is equivalent to a unique block matrix of companion matrices associated to polynomials that divide each other in the manner described above. We call $a_1(x), \ldots, a_m(x)$ also the invariant factors of $A$.

1) Prove that two matrices $A$ and $B$ in $M_n(F)$ are similar, i.e. there is an invertible matrix $C \in M_n(F)$ such that $CAC^{-1} = B$, if and only if $A$ and $B$ has the same invariant factors. (This is quite easy.)

(2) Let $F$ be a finite field with $q$ elements; let $\text{GL}_n(F)$ act on $M_n(F)$ by $(C, A) \mapsto CAC^{-1}$. Write a formula for the number of orbits of this action for $n = 1, 2, 3, 4, 5, 6$.

Guidance: I don’t think the Cauchy-Frobenius formula is of any help in this case. I suggest using the statement in (1). After doing those cases (you can explain in detail the cases $n = 2, 3$ and just compute the rest) you’ll be able to write a general “formula” that holds for every $n$. 
