1) Let $R$ be a commutative ring with 1. An ideal $I \triangleleft R$ is called nilpotent if there is a positive integer $k$ such that $I^k = 0$. Recalling the definition of a product of ideals, we see that $I$ is nilpotent with $I^k = 0$ if and only if for any $k$ elements $x_1, \ldots, x_k$ of $I$ we have $x_1 x_2 \cdots x_k = 0$. Let $R_0$ be any commutative ring and $J$ an ideal of $R_0$. Let $R = R_0 / J^k$ and let $I$ be the ideal which is the image of $J$ under $R_0 \rightarrow R_0 / J^k$. Then $I$ is nilpotent and $I^k = 0$. As a concrete example, the ideal generated by $x$ in $F[x]/(x^k)$ is nilpotent (of degree $k$).

Let $I$ be a nilpotent ideal of $R$. Let $M, N$ be $R$-modules and $f : M \rightarrow N$ an $R$-module homomorphism. Prove that there is a well defined induced homomorphism of $R$-modules $\bar{f} : M/IM \rightarrow N/IN$. Prove that if $\bar{f}$ is surjective then $f$ is surjective.

2) Let $V$ be a finite dimensional vector space over $F$ and $T : V \rightarrow V$ a linear transformation by which we consider $V$ as an $F[x]$-module. Prove that $V$ is a cyclic module if and only if the minimal polynomial of $T$ is equal to its characteristic polynomial.

3) Let $R$ be an integral domain and let $f : M \rightarrow M_1$ be a surjective homomorphism with kernel $M_2$. Prove that

$$\text{rank}(M) = \text{rank}(M_1) + \text{rank}(M_2).$$

(You can consult the exercises in Dummit and Foote, pp. 468-469 about how to solve some difficulties that arise in the proof.)