Algebra 4 (2004-05) – Assignment 2

Instructor: Dr. Eyal Goren

Submit by Monday, January 24, 24:00 by mail-box on 10^{th} floor.

1) Let R be a commutative ring with 1. An ideal $I \lhd R$ is called *nilpotent* if there is a positive integer k such that $I^k = 0$. Recalling the definition of a product of ideals, we see that I is nilpotent with $I^k = 0$ if and only if for any k elements $x_1, \dots, x_k$ of I we have $x_1 x_2 \cdots x_k = 0$. Let R_0 be any commutative ring and J an ideal of R_0. Let $R = R_0/J^k$ and let I be the ideal which is the image of J under $R_0 \longrightarrow R_0/J^k$. Then I is nilpotent and $I^k = 0$. As a concrete example, the ideal generated by x in $F[x]/(x^k)$ is nilpotent (of degree k).

Let I be a nilpotent ideal of R. Let M, N be R-modules and $f : M \longrightarrow N$ an R-module homomorphism. Prove that there is a well defined induced homomorphism of R-modules $\bar{f} : M/IM \longrightarrow N/IN$. Prove that if $\bar{f}$ is surjective then f is surjective.

2) Let V be a finite dimensional vector space over $\mathbb{F}$ and $T : V \longrightarrow V$ a linear transformation by which we consider V as an $\mathbb{F}[x]$-module. Prove that V is a cyclic module if and only if the minimal polynomial of T is equal to its characteristic polynomial.

3) Let R be an integral domain and let $f : M \longrightarrow M_1$ be a surjective homomorphism with kernel M_2. Prove that

$$\mathrm{rank}(M) = \mathrm{rank}(M_1) + \mathrm{rank}(M_2).$$

(You can consult the exercises in Dummit and Foote, pp. 468-469 about how to solve some difficulties that arise in the proof.)