Algebra 4 (2004-05) – Assignment 10

Instructor: Dr. Eyal Goren

Submit by Monday, April 11, 12:00.

1) Let $K \supset L \supset \mathbb{Q}$ be field such that $[K : L] = [L : \mathbb{Q}] = 2$. Assume that K is not Galois over $\mathbb{Q}$. (i) Prove that L is the unique proper subfield of K. (ii) Write $K = L(\sqrt{d}), d \in L$ and conclude $K = \mathbb{Q}(\sqrt{d})$. (iii) Prove that there is a field $M \supset K$ such that $M/\mathbb{Q}$ is Galois with Galois group isomorphic to D_8 (the dihedral group with 8 elements). This can be done using the minimal polynomial of $\sqrt{d}$ over $\mathbb{Q}$. (iv) Provide an example of such a field K. (Cf. Exe. 11, page 617 in Dummit and Foote, but note that only the results we proved in class are needed to solve the question).

2) The *Inverse Galois Problem* ask if for every finite group G there is a Galois extension $K/\mathbb{Q}$ with $\mathrm{Gal}(K/\mathbb{Q}) \cong G$. This is a wide open problem. Experts believe that the answer is yes.

Prove that if the order of G is less than 10 and G is not the quaternion group of order 8 then such an extension exists. (If you'd like to attempt Q, you can take as a guidance Dummit and Foote, Ex. 27, p. 584).

3) Show that $\mathbb{Q}(\sqrt{2+\sqrt{2}})/\mathbb{Q}$ is Galois with Galois group isomorphic to $\mathbb{Z}/4\mathbb{Z}$.