Algebra 4 (2004-05) – Assignment 1

Instructor: Dr. Eyal Goren

Submit by Monday, January 17, 24:00 by mail-box on 10^{th} floor.

1) Prove the Chinese remainder theorem for modules: Let R be a commutative ring with 1, let $I_1, \dots, I_k$ be relatively prime ideals of R and let M be an R-module. Let $J = I_1 I_2 \cdots I_k$. Then

$$M/JM \cong M/I_1M \oplus M/I_2M \oplus \cdots \oplus M/I_kM.$$

Consider now the application of this theorem to vector spaces, viewed as $\mathbb{F}[x]$-modules. Let V be a vector space and $T : V \to V$ a linear transformation. Consider the decomposition of its minimal polynomial $m_T(x)$ into relatively prime elements: $m_T(x) = m_1(x)^{a_1} \cdots m_k(x)^{a_k}$. What does the Chinese Remainder Theorem gives in this case? (Explain also how to get the invariant subspaces $V_1, \dots, V_k$ by a projection $V \to V_i$).

Demonstrate everything explicitly for the vector space $\mathbb{C}^5$ and the matrix

$$\begin{pmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 2 \end{pmatrix}.$$

2) Let R be a commutative ring with 1. Prove that if $R^m \cong R^n$ as R-modules then $m = n$.
Suggestion: try and reducing everything modulo a suitable ideal and use the theory of vector spaces.