1) Prove the Chinese remainder theorem for modules: Let $R$ be a commutative ring with 1, let $I_1, \ldots, I_k$ be relatively prime ideals of $R$ and let $M$ be an $R$-module. Let $J = I_1 I_2 \cdots I_k$. Then

$$M/JM \cong M/I_1M \oplus M/I_2M \oplus \cdots \oplus M/I_kM.$$ 

Consider now the application of this theorem to vector spaces, viewed as $\mathbb{F}[x]$-modules. Let $V$ be a vector space and $T : V \to V$ a linear transformation. Consider the decomposition of its minimal polynomial $m_T(x)$ into relatively prime elements: $m_T(x) = m_1(x)^{a_1} \cdots m_k(x)^{a_k}$. What does the Chinese Remainder Theorem give in this case? (Explain also how to get the invariant subspaces $V_1, \ldots, V_k$ by a projection $V \to V_i$).

Demonstrate everything explicitly for the vector space $\mathbb{C}^5$ and the matrix

$$
\begin{pmatrix}
1 & 1 & 0 & 0 & 0 \\
0 & 1 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 2 & 0 \\
0 & 0 & 0 & 0 & 2
\end{pmatrix}.
$$

2) Let $R$ be a commutative ring with 1. Prove that if $R^m \cong R^n$ as $R$-modules then $m = n$.

Suggestion: try and reducing everything modulo a suitable ideal and use the theory of vector spaces.