Algebra 3 (2003-04) – Assignment 9

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Submit by Monday, November 17, 12:00 by mail-box on 10^{th} floor.

1.

(1) Let R, S be two rings. Show that $R \times S$ (also denoted $R \oplus S$) is a ring under the operations

$$(r_1, s_1) + (r_2, s_2) = (r_1 + r_2, s_1 + s_2), \qquad (r_1, s_1)(r_2, s_2) = (r_1 r_2, s_1 s_2).$$

(2) Find the ideals of $\mathbb{Z}$, $\mathbb{Z} \oplus \mathbb{Z}$.

2. Let R be a commutative ring.

(1) Find the units of $R[x], R[[x]], R((x))$.
(2) Assume that R is a field. Prove that $R((x))$ is isomorphic to $\text{Quot}(R[[x]])$.
(3) Prove that $\text{Quot}(\mathbb{Z}[[x]])$ is properly contained in $\mathbb{Q}((x))$.

3. Let R be a ring. An element $x \in R$ is called nilpotent if $x^m = 0$ for some positive integer m.[1]

(1) Find the nilpotent elements of $\mathbb{Z}/n\mathbb{Z}$ in terms of the prime factorization of n.
(2) Prove that if x is nilpotent then $1 + x$ is a unit.
(3) Assume that R is commutative. Prove that the set of nilpotent elements is an ideal of R. It is called the nil-radical of R and is denoted by $\text{nil}(R)$, or $\sqrt{(0)}$.
(4) Prove that $\text{nil}(R/\text{nil}(R)) = \{0\}$.

Bonus. Find a ring R with elements $x, y \in R$ such that $xy = 1$ but $yx \neq 1$.

[1]Note that if $\mathbb{F}$ is a field, $R = M_n(\mathbb{F})$ then the nilpotent elements of R are the nilpotent matrices, which can be also characterized as the matrices with characteristic polynomial t^n.