Algebra 3 (2003-04) – Assignment 7

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Submit by Monday, October 27, 12:00 by mail-box on 10^{th} floor.
For full marks do questions 2, 4 and one of questions 1, 3. Solving all questions is highly recommended.

1)

(1) Given a positive integer N prove that there are finitely many groups of order N up to isomorphism.
(2) Given a positive integer N prove that there are finitely many finite groups with N conjugacy classes. You may use the following fact (which is not difficult to prove!): Given $n > 0$ and a rational number q there are only finitely many n-tuples $(c_1, \dots, c_n)$ of natural numbers such that $q = \frac{1}{c_1} + \dots + \frac{1}{c_n}$.
(3) Find the groups in the preceding question for $N = 1, 2, 3$. (Prove your answer; you may use the classification of groups of small order we gave in the past.)

2)

(1) Let G be a group, then $G/Z(G)$ is not a non-trivial cyclic group. (Rephrased: If $G/Z(G)$ is cyclic then it is the trivial group of one element.)
(2) Let p be a prime. Prove that a group of order p^2 is abelian.

3) Prove the following facts concerning free groups:[1]

(1) given a group G, and d elements $s_1, \dots s_d$ in G, there is a unique group homomorphism $f : \mathscr{F}(d) \longrightarrow G$ such that $f(x_i) = s_i$;
(2) if G is a group generated by d elements there is a surjective group homomorphism $\mathscr{F}(d) \longrightarrow G$;
(3) if $w_1, \dots w_r$ are words in $\mathscr{F}(d)$, let N be the minimal normal subgroup containing all the w_i (prove such an N exists!). The group $\mathscr{F}(d)/N$ is also denoted by $< x_1, \dots, x_d | w_1, \dots, w_r >$ and is said to be given by the generators $x_1, \dots x_d$ and relations $w_1, \dots, w_r$.
(4) Prove that $\mathbb{Z}/n\mathbb{Z} \cong < x_1 | x_1^n >$, $\mathbb{Z}^2 \cong < x_1, x_2 | x_1 x_2 x_1^{-1} x_2^{-1} >$.
(5) if $d = 1$ then $\mathscr{F}(d) \cong \mathbb{Z}$ but if $d > 1$ then $\mathscr{F}(d)$ is a non-commutative infinite group.

4)

(1) Let p be a prime. Find all p-Sylow subgroups of S_p. How many are there?
(2) Find for every prime p how many p-Sylow subgroups S_5 has; write explicitly one p-Sylow subgroup for each prime.

[1] We let $\mathscr{F}(d)$ be the free group on the generators $x_1, \dots, x_d$. The first three claims were proven in class and there is no need to repeat the proof.