Algebra 3 (2003-04) – Assignment 5

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Submit by Monday, October 13, 12:00 by mail-box on 10^{th} floor. Solve at least 5 questions for full marks.

1) Let p be a prime. Let G be a finite p-group ($|G| = p^a$ for some a). Let V be a finite dimensional vector space over the field with p-elements $\mathbb{F}_p = \mathbb{Z}/p\mathbb{Z}$. Prove that there is a non-zero vector $v \in V$ such that every $g \in G$ fixes v.

Remark: Note that if V is a vector space over $\mathbb{F}_q$, where $q = p^b$, and is of finite dimension, then we may also view V as a finite dimensional vector space over $\mathbb{F}_p$.

2) Let G be a finite group of order pN. Assume that p is the minimal prime dividing the order of G. Prove that every subgroup of index p is normal.

3) Find how many different designs there are of necklaces with 3 diamonds, 4 rubies and 5 sapphires.

4) Let A be a proper subgroup of a finite group G. Prove that $G \neq \cup_{g \in G} gAg^{-1}$. Prove that this statement may fail for infinite groups (suggestion: try $\mathrm{GL}_2(\mathbb{C})$).

5) Consider the permutations of the set $X = \{1, 2, 3, \dots\}$ and the subgroup G consisting of even permutations of "finite support", i.e., even permutations that move only finitely many elements (for such permutation the notion of a sign is well defined). Prove that G is an infinite simple group.

6) Let G be a group and $\mathrm{Aut}(G)$, the *automorphism group of* G, the set of group isomorphisms $f : G \longrightarrow G$. Prove that $\mathrm{Aut}(G)$ is a group under composition.

Given an element $g \in G$ define a function

$$f_g : G \longrightarrow G, \quad f_g(a) = gag^{-1}.$$

Prove that this gives a homomorphism $G \longrightarrow \mathrm{Aut}(G)$ with kernel $Z(G)$. The group $G/Z(G)$ is called the group of *inner automorphisms*, denoted $\mathrm{Inn}(G)$, and the group $\mathrm{Out}(G) := \mathrm{Aut}(G)/\mathrm{Inn}(G)$ is called the group of *outer automorphisms*.

Calculate the groups $\mathrm{Aut}(G), \mathrm{Inn}(G)$ and $\mathrm{Out}(G)$ for G : (i) a finite cyclic group; (ii) S_3.