Algebra 3 (2003-04) – Assignment 4

Instructor: Dr. Eyal Goren

Submit by Monday, October 6, 12:00 by mail-box on 10^{th} floor.

1) If G, H are finite groups such that $(|G|, |H|) = 1$ then every group homomorphism $f : G \to H$ is trivial ($f(G) = \{1\}$).

2) Let $f : G \to H$ be a group homomorphism. Let $N \triangleleft G$ be a subgroup such that $N \subseteq \text{Ker}(f)$. Show that there is a unique homomorphism $f' : G/N \to H$ such that $f' \circ \pi_N = f$, where $\pi_N : G \to G/N$ is the canonical homomorphism $\pi_N(g) = gN$ - we say that f *factors through* G/N. Moreover, $\text{Ker}(f') = \text{Ker}(f)/N$.

Conclude that any homomorphism $G \to H$ of G into a commutative group H factors through G/G'.

3) Find all possible homomorphisms $Q \to S_3$.

4) Let p be a prime and let G be a group of order $p^a m$, with $(p, m) = 1$. Let P be a subgroup of G of order p^a. Let N be a normal subgroup of G. Prove that if the order of N is $p^b n$ with $(p, n) = 1$ then $|P \cap N| = p^b$. Prove also that $|PN/N| = p^{a-b}$.

Remark: P is called a p-Sylow subgroup of G. That means that its order is the exact power of p dividing the order of G. In fact, it is remarkable that such a subgroup always exists. This is the content of one of Sylow's theorem we'll soon prove. The exercise shows that the intersection of a p-Sylow subgroup of G with a normal subgroup N is a p-Sylow subgroup of N.