Algebra 3 (2003-04) – Assignment 11

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Submit by Monday, December 1, 12:00 by mail-box on 10^{th} floor.

We use the notation $\mathbb{F}_q$ to denote a finite field with q elements. If q is a prime then $\mathbb{F}_q \cong \mathbb{Z}/q\mathbb{Z}$.

1. Prove that $\mathbb{F}_3[x]/(x^2+1)$ is a field with 9 elements. Find the inverse of $x+2$ in this field (for that use the Euclidean algorithm to find the g.c.d. of $x+2$ and x^2+1). Prove that $\mathbb{F}_5[x]/(x^2+1)$ is not a field.

2. Give an example of an integral domain R and two elements $x, y \in R$ such that there is a g.c.d. for x and y but the ideal (x, y) is not principal.

3. Show that the ideal $(-1+3i, 1+5i)$ in $\mathbb{Z}[i]$ is principal and find a generator for it.

4. Making use of the identity $(1+\sqrt{-5})(1-\sqrt{-5}) = 2 \cdot 3$, prove that the ring $\mathbb{Z}[\sqrt{-5}]$ is not a UFD.

5. For a UFD R define the least common multiple $l.c.m.(x, y)$ of any two elements $x, y \in R$. Prove it is well defined and satisfies properties analogous (but "opposite") to those of the greatest common divisor. Find a formula for $l.c.m.(x, y)$ in terms of the prime factorization of x and y.