Algebra 3 (2003-04) – Assignment 1

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Submit by Monday, September 15, 12:00 by mail-box on 10th floor.

1) Let $G$ be a finite group such that for every $g \in G$ we have $g^2 = e$.
   (1) Prove that $G$ is an abelian group.
   (2) Prove that $G$ has $2^n$ elements.

2) Let $G$ be a finite group of even order. Prove that $G$ has an element of order 2. (Hint: consider the association $g \mapsto g^{-1}$).

3) Let $p$ be a prime number. Prove that $(p - 1)! \equiv -1 \pmod{p}$. (Hint: for a finite abelian group $G$ consider the product $\Pi_{g \in G} g$).

4) Write down all the elements of $GL_2(\mathbb{F}_2)$. Consider the action of this group on the set of non-zero vectors in $\mathbb{F}_2^2$ (the two dimensional vector space over $\mathbb{F}_2$). Show that this allows one to identify the group $GL_2(\mathbb{F}_2)$ with the symmetric group $S_3$.

5) Consider the upper triangular $3 \times 3$ invertible matrices with entries in $\mathbb{F}_2$. How many elements does this group have? Is it abelian? Find the order of each of its elements.

Bonus: identify this group with a more familiar group of order 8.