Algebra 3 (2003-04) – Assignment 1

Instructor: Dr. Eyal Goren

Submit by Monday, September 15, 12:00 by mail-box on 10^{th} floor.

1) Let G be a finite group such that for every $g \in G$ we have $g^2 = e$.

(1) Prove that G is an abelian group.
(2) Prove that G has 2^n elements.

2) Let G be a finite group of even order. Prove that G has an element of order 2. (Hint: consider the association $g \mapsto g^{-1}$).

3) Let p be a prime number. Prove that $(p-1)! \equiv -1 \pmod{p}$. (Hint: for a finite abelian group G consider the product $\Pi_{g \in G}\, g$).

4) Write down all the elements of $\mathrm{GL}_2(\mathbb{F}_2)$. Consider the action of this group on the set of non-zero vectors in $\mathbb{F}_2^2$ (the two dimensional vector space over $\mathbb{F}_2$). Show that this allows one to identify the group $\mathrm{GL}_2(\mathbb{F}_2)$ with the symmetric group S_3.

5) Consider the upper triangular 3×3 invertible matrices with entries in $\mathbb{F}_2$. How many elements does this group have? Is it abelian? Find the order of each of its elements.

Bonus: identify this group with a more familiar group of order 8.