Algebra 3 (2004-05) – Assignment 9

Instructor: Dr. Eyal Goren

Submit by Monday, November 22, 17:00 by mail-box on 10^{th} floor.

1).

(1) Let R be a commutative ring. Prove that $R[\![x]\!]^{\times} = \{\sum_{n=0}^{\infty} a_n x^n : a_n \in R, a_0 \in R^{\times}\}$.

(2) Find all the ideals of the ring $\mathbb{C}[\![x]\!]$.

2). Let k be a field. Find all left and right ideals of the ring $M_n(k)$.

3).

(1) Let R, S be two rings. Show that $R \times S$ (also denoted $R \oplus S$) is a ring under the operations $(r_1, s_1) + (r_2, s_2) = (r_1 + r_2, s_1 + s_2)$, $(r_1, s_1)(r_2, s_2) = (r_1 r_2, s_1 s_2)$.

(2) Prove that every left ideal of $R \oplus S$ has the form $I \times J$, where $I \lhd R, J \lhd S$ are left ideals.

(3) Make this explicit for $R = S = \mathbb{Z}$. Exhibit a *subgroup* of $\mathbb{Z} \oplus \mathbb{Z}$ which is not of this form (hence not an ideal).

4).

(1) Show that $\mathbb{Z}[i]/(2 + 3i)$ is a finite field. How many elements does it have?

(2) Show that $\mathbb{Z}[i]/(3)$ is not a field.

5). Prove that in the ring $\mathbb{Q}[x, y]$ the ideal (x, y) is not principal.