Algebra 3 (2004-05) – Assignment 7

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Submit by Monday, November 1, 17:00 by mail-box on 10th floor. Answer question no. 5 and at least 4 other questions. Every additional question is a bonus.

1). How many elements of order 7 does a simple group of order 168 must have?

2). It is known that $GL_3(\mathbb{F}_2)$ is a simple group of order 168, and in fact is the unique one up to isomorphism. Show, at least, that none of its Sylow groups is normal.

3). Prove that a group of order 36 has a normal Sylow subgroup.

4). Prove that in a group of order 231 the center contains the 11-Sylow and that the 7-Sylow is normal.


6). Let $G$ be a group. Prove that there is a group homomorphism 

$$G \rightarrow \text{Aut}(G), \quad g \mapsto \phi_g,$$

where $\phi_g(x) = gxg^{-1}$, whose kernel is $Z(G)$. The image of this homomorphism is denoted $\text{Inn}(G)$ (the inner automorphisms of $G$). Prove that the image is a normal subgroup of $\text{Aut}(G)$. The quotient $\text{Aut}(G)/\text{Inn}(G)$ is denoted $\text{Out}(G)$ and is called the outer automorphism group of $G$.

7). In this exercise we shall prove that $\text{Aut}(S_n) = S_n$ for $n > 6$. (The results holds true for $n > 3$ and fails for $n = 6$.)

(1) Prove that an automorphism of $S_n$ takes an element of order 2 to an element of order 2.

(2) For $n \neq 6$ use an argument involving centralizers to show that an automorphism of $S_n$ takes a transposition to a transposition.

(3) Prove that every automorphism has the effect $(12) \mapsto (ab_2),(13) \mapsto (ab_3),\dots,(1n) \mapsto (ab_n)$, for some distinct $a,b_2,\ldots,b_n \in \{1,2,\ldots,n\}$. Conclude that $\sharp \text{Aut}(S_n) \leq n!$.

(4) Show that for $n > 6$ there is an isomorphism $S_n \cong \text{Aut}(S_n)$.

8). Find the number of $p$-Sylow subgroups of $S_5$ and write one such subgroup for each prime.

9). Find a $p$-Sylow subgroup of $S_{2p}$. Find all Sylow subgroups of $D_{12}$.

10). For every prime $p$ dividing its order write an example of a $p$-Sylow subgroup and determine their number for the group $SL_2(\mathbb{F}_3)$.

11). Prove that a group of order $pqr$, $p < q < r$ primes, is not simple.

12). Prove that a group $G$ of order 36, 40, 48, 54 is not simple.

13). Let $G \triangleleft H$ then conjugation by elements of $H$ induce automorphisms of $G$; if $h \in H$ we have $x \mapsto hxh^{-1}$ is an automorphism $\tau$ of $G$. In that case we say $\tau$ is induced by conjugation inside $H$.

Given $G$, prove that there is a group $H$ such that $G \triangleleft H$ and every automorphism of $G$ is induced by conjugation inside $H$.

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