Algebra 3 (2004-05) – Assignment 7

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Submit by Monday, November 1, 17:00 by mail-box on 10^{th} floor. Answer question no. 5 and at least 4 other questions. Every additional question is a bonus.

1). How many elements of order 7 does a simple group of order 168 must have?

2). It is known that $GL_3(\mathbb{F}_2)$ is a simple group of order 168, and in fact is the unique one up to isomorphism. Show, at least, that none of its Sylow groups is normal.

3). Prove that a group of order 36 has a normal Sylow subgroup.

4). Prove that in a group of order 231 the center contains the 11-Sylow and that the 7-Sylow is normal.

5). Let $p > 2$ be a prime. Using semi-direct products construct two non-isomorphic non-abelian groups of order p^3.

6). Let G be a group. Prove that there is a group homomorphism

$$G \longrightarrow \mathrm{Aut}(G), \qquad g \mapsto \phi_g,$$

where $\phi_g(x) = gxg^{-1}$, whose kernel is $Z(G)$. The image of this homomorphism is denoted $\mathrm{Inn}(G)$ (the inner automorphisms of G). Prove that the image is a normal subgroup of $\mathrm{Aut}(g)$. The quotient $\mathrm{Aut}(G)/\mathrm{Inn}(G)$ is denoted Out(G) and is called the outer automorphism group of G.

7). In this exercise we shall prove that $\mathrm{Aut}(S_n) = S_n$ for $n > 6$. (The results holds true for $n > 3$ and fails for $n = 6$.)

(1) Prove that an automorphism of S_n takes an element of order 2 to an element of order 2.
(2) For $n \neq 6$ use an argument involving centralizers to show that an automorphism of S_n takes a transposition to a transposition.
(3) Prove that every automorphism has the effect $(12) \mapsto (ab_2), (13) \mapsto (ab_3), \dots, (1n) \mapsto (ab_n)$,for some distinct $a, b_2, \dots, b_n \in \{1, 2, \dots, n\}$. Conclude that $\sharp\, Aut(S_n) \leq n!$.
(4) Show that for $n > 6$ there is an isomorphism $S_n \cong \mathrm{Aut}(S_n)$.

8). Find the number of p-Sylow subgroups of S_5 and write one such subgroup for each prime.

9). Find a p-Sylow subgroup of S_{2p}. Find all Sylow subgroups of D_{12}.

10). For every prime p dividing its order write an example of a p-Sylow subgroup and determine their number for the group $\mathrm{SL}_2(\mathbb{F}_3)$.

11). Prove that a group of order pqr, $p < q < r$ primes, is not simple.

12). Prove that a group G of order $36, 40, 48, 54$ is not simple.

13). Let $G \lhd H$ then conjugation by elements of H induce automorphisms of G; if $h \in H$ we have $x \mapsto hxh^{-1}$ is an automorphism τ of G. In that case we say τ is induced by conjugation inside H.

Given G, prove that there is a group H such that $G \lhd H$ and every automorphism of G is induced by conjugation inside H.