

Algebra 3 (2004-05) – Assignment 6

Instructor: Dr. Eyal Goren

Submit by Monday, October 25, 17:00 by mail-box on 10th floor.

1)

- (1) Let G be a group. Prove that $G/Z(G)$ cannot be a non-trivial cyclic group (namely, if it is a cyclic group it is the trivial group of one element $\{1\}$).
- (2) Apply that to prove that any group of order p^2 , p a prime, is abelian.
- (3) What can be the size of the center of groups G of order p^3 ? What is then the structure of $G/Z(G)$?
- (4) Verify the general answer you are giving by discussing the case of the group $U \subset \text{GL}_3(\mathbb{Z}/p\mathbb{Z})$, consisting of matrices of the form $\begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix}$.

2) Let G be a finite p -group and let $\{1\} \neq H \triangleleft G$. Prove that $H \cap Z(G) \neq \{1\}$.

3) Prove that $\mathbb{Z}^2 \cong \langle x, y | xyx^{-1}y^{-1} \rangle$.

4) Prove that $D_{2n} = \langle x, y | x^n, y^2, yxyx \rangle$.

5) **Bonus question.** Is the group $\langle x, y | x^2, y^3 \rangle$ finite or infinite?