Algebra 3 (2004-05) – Assignment 10

Instructor: Dr. Eyal Goren

Submit by Wednesday, December 1 , 17:00 by mail-box on 10^{th} floor.

1). Prove that $\mathbb{Z}[x]$ is not a PID. Prove that every ideal of $\mathbb{Z}[x]$ is of the form $\langle n \rangle$ for some integer n, $\langle f(x) \rangle$ for some polynomial $f(x) \in \mathbb{Z}[x]$ or $\langle n, f(x) \rangle$. In each case provide an example where such an ideal is prime.

2). Prove that $\mathbb{Z}[\omega]$ is an Euclidean ring, where $\omega = e^{2\pi i/3}$ (include a proof that $\mathbb{Z}[\omega]$ is a ring).

3). Use the Euclidean algorithm to find a generator for the ideal $(1+3i, 2)$ in $\mathbb{Z}[i]$. Prove that $\mathbb{Z}[i]/(1+2i)$ is a field. Find the multiplicative inverse of $2 + 3i$ in it.

4). Find an integer x satisfying

$$x \equiv 2 \pmod 5, \quad x \equiv 1 \pmod{12}, \quad x \equiv 5 \pmod{13}.$$

5). Let p be a prime. Prove that there are finite fields of p^2 and p^3 elements.