ASSIGNMENT 6 - ALGEBRAIC GEOMETRY 189-706 A

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To be submitted Monday, October 25.

1. Prove that $\alpha \in \mathbb{Q}(i)$ is integral over $\mathbb{Z}[i]$ if and only if $\alpha \in \mathbb{Z}[i]$.

2. Prove that $N_{\mathbb{Q}(\sqrt{5})}(\mathbb{Z}) = \mathbb{Z}\left[\frac{1+\sqrt{5}}{2}\right]$.

3. Let U be a multiplicative set in A. Then $A \subset B$ is an integral extension implies $A[U^{-1}] \subset B[U^{-1}]$ is an integral extension.

4. $A \subset B$ is an integral extension, $\mathfrak{m} \lhd B$, $\mathfrak{n} = \mathfrak{m} \cap A$. Then $A/\mathfrak{n} \subset B/\mathfrak{m}$ is an intergral extension.

5. Let $A \subset B$ be an integral extension. Assume that A and B are domains. Then, we proved, A is a field iff B is a field. What is the geometric content of this statement?

6. Give an example of rings $A \subset B$ such that: (i) $\mathfrak{q} \lhd B$ is maximal and $\mathfrak{q} \cap A$ is not. (ii)$\mathfrak{q} \lhd B$, $\mathfrak{q} \cap A$ is maximal and $\mathfrak{q}$ is not.

7. Let A be a ring and U a multiplicative set in A. Let M be a module over A and $M_1, M_2 \subset M$ sub modules. Prove

$$M_1[U^{-1}] \cap M_2[U^{-1}] = (M_1 \cap M_2)[U^{-1}].$$

8. Let $f : X \longrightarrow Y$ be a surjective morphism of affine varieties over an algebraically closed field k. Let $A = \mathcal{O}(Y)$ and $B = \mathcal{O}(X)$. Let $\varphi = f^*$. Give a geometric proof that given a prime ideal $\mathfrak{p}$ of A there is a prime ideal $\mathfrak{q}$ of B such that $\varphi^{-1}\mathfrak{q} = \mathfrak{p}$.