ASSIGNMENT 10 - ALGEBRAIC GEOMETRY 189-706 A

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To be submitted Wednesday, November 24.

1. Let Z be an irreducible closed set of a scheme S. A point $z \in Z$ is called a *generic point* of Z if $Z = \overline{\{z\}}$.

a. Prove that z is a generic point if and only if every open subset U of Z contains z.

b. Let $S = \mathrm{Spec}(R)$ be a scheme. Let $s \in S$. Prove that $\overline{\{s\}}$ is an irreducible closed set and s is a generic point.

c. Prove that every irreducible closed set $Z \subset \mathrm{spec}(R)$ has a generic point. In fact, show that $Z = V(\mathfrak{p})$ for some prime ideal $\mathfrak{p} \lhd R$ and that $\mathfrak{p}$ is such a generic point.

d. Let Z be an irreducible closed set of a scheme S. Prove that S has a unique generic point.

2. Consider a topological space with three points. For every possible topology determine whether this space is isomorphic to $\mathrm{spec}(R)$ for some ring R. Prove first that the topological space $\mathrm{spec}(R_1) \coprod \mathrm{spec}(R_2)$ is isomorphic to $\mathrm{spec}(R_1 \oplus R_2)$. (In fact, also as schemes, but I leave that to you). Prove also that for every two points in an affine scheme there exists an open set containing one of the points and not the other. (That is the T_0 property).

3. Solve questions 2.5, 2.12 in Hartshorne p.80.

4. Find a toric varieties isomorphic to $\mathbb{P}^3$, and to $\mathbb{C} \times \mathbb{P}^1$.