Alon-Boppana Lower Bound
and
Ramanujan Graphs

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(This lecture is based on [HLW06])

Recall that if $G$ is an undirected graph with $n$ vertices, then its adjacency matrix has $n$ real eigenvalues $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n$. Then, in order to construct a family of $d$-regular expanders, by the Alon-Milman theorem, we need to bound the spectral gap $(d - \lambda_2)$ from below. Hence it is important to understand the behavior of $\lambda_2$.

1 Main statement and definition

**Theorem 1.1** (Alon-Boppana). There exists a constant $c$ such that for every connected finite regular graph $G$,

$$\lambda_2(G) \geq 2\sqrt{d-1} \left( 1 - \frac{c}{\Delta^2} \right)$$

where $\Delta = \text{diam}(G)$ and $d = \text{deg}(v)$ for every vertex $v$.

**Corollary 1.2.** Let $(G_m)_{m=1}^\infty$ be a family of connected, $d$-regular, finite graphs with $|V(G_m)| \to \infty$ as $m \to \infty$. Then,

$$\liminf_{m \to \infty} \lambda_2(G_m) \geq 2\sqrt{d-1}$$
In view of this corollary, we define Ramanujan graphs as graphs that are optimal in this sense:

**Definition 1.3.** An \((n,d)\)-graph \(G\) \((n\) vertices and \(d\)-regular) is called **Ramanujan** if

\[ \lambda(G) \leq 2\sqrt{d-1} \]

where \(\lambda(G) = \max_{|\lambda| \neq d} |\lambda|\).

## 2 The infinite tree \(T_d\) and its spectrum

Throughout this section \(T = T_d, V = V(T)\) and \(N(v)\) denotes the set of neighbors of a vertex \(v \in V\). We can define

\[ A_T : l_2(V) \to l_2(V) \]

just like in the finite case, that is,

\[ (A_T f)(v) = \sum_{w \in N(v)} f(w). \]

We view \(A_T\) in \(B(l_2(V))\), the Banach algebra of bounded linear operators on \(l_2(V)\).

**Definition 2.1.** We say a function \(f : V \to \mathbb{C}\) is **spherical around vertex** \(v\) if \(f(u)\) depends only on the distance between \(u\) and \(v\) (\(\text{dist}(u,v)\)).

For any function \(f : V \to \mathbb{C}\), we can define its **spherical symmetrization around** \(v\) to be a function \(\tilde{f}\) that is spherical around \(v\) and such that

\[ \sum_{\text{dist}(u,v) = i} \tilde{f}(u) = \sum_{\text{dist}(v,u) = i} f(u) \text{ for every } i \geq 0. \]

**Definition 2.2.** The **spectrum** of \(A_T\) is

\[ \sigma(A_T) := \{ \lambda : \lambda I - A_T \text{ is not invertible} \} \]

(For basic properties of the spectrum, see [Rud91])

**Theorem 2.3** (Cartier). \(\sigma(A_T) = [-2\sqrt{d-1}, 2\sqrt{d-1}]\)
Proof. (sketch)

We start by fixing a vertex \( v \in V \) and consider it to be the ‘root’ of our tree.

It can show that in our case,
\[ \lambda \in \sigma(A_T) \iff \delta_v \notin \text{img}(\lambda I - A_T) \]
where \( \delta_v \) is the characteristic function of \( v \) \((\delta_v(v) = 1 \text{ and } \delta_v(u) = 0 \text{ for all the other } u \in V)\).

So, it is enough to show that
\[ \delta_v = (\lambda I - A_T) \cdot f \tag{1} \]
has a solution (in \( l_2(V) \)) if \( |\lambda| < 2\sqrt{d-1} \) and does not have a solution if \( |\lambda| > 2\sqrt{d-1} \) (see theorems 12.26 and 10.13 in [Rud91]).

**Claim 1.** We may assume \( f \) in equation (1) is spherical around \( v \) (more precisely, if (1) has a solution for some \( f \), then it also has a solution for some \( \tilde{f} \) spherical around \( v \)).

In fact, if \( f \) satisfies (1), then it is easy to show its spherical symmetrization around \( v \) also satisfies (1).

If \( f \) is spherical around \( v \), then it is determined by a sequence \( f_0, f_1, f_2, \ldots \) such that \( f(u) = f_i \) for every \( u \) satisfying \( \text{dist}(u,v) = i \). Using this notation it is not hard to see that a spherical function \( f \) satisfies (1) if and only if:
\[ \lambda f_0 = df_1 + 1 \]
\[ \lambda f_i = f_{i-1} + (d-1)f_{i+1} \quad \text{for } i \geq 1 \tag{2} \]

Using linear algebra we can show the solutions \( \{f_i\} \) of (2) are of the form \( f_i = \alpha \rho_i^+ + \beta \rho_i^- \), where \( \rho_{\pm} = \frac{\lambda \pm \sqrt{\lambda^2 - 4(d-1)}}{2(d-1)} \).

Now, if \( |\lambda| < 2\sqrt{d-1} \) then \( |\rho_{\pm}| = \frac{1}{\sqrt{d-1}} \). Hence, \( |f_i| = \Theta((d-1)^{-\frac{i}{2}}) \) [according to the authors, this is an easy computation; the upper bound is easy to check but I could not verify the lower bound]. Since there are \( \Theta((d-1)^i) \) vertices at distance \( i \), this means such an \( f \) would not be in \( l_2 \) (in fact, \( ||f||_2^2 \geq C \sum_{i=0}^{\infty} (d-1)^i)((d-1)^{-i/2})^2 = C \sum_{i=0}^{\infty} 1 = \infty \)). Hence, if \( |\lambda| < 2\sqrt{d-1} \), then \( \lambda \notin \sigma(A_T) \).
If $\lambda > 2\sqrt{d-1}$, then $r := |\rho_-| < \frac{1}{\sqrt{d-1}}$. In this case, we choose $\alpha = 0$, giving $f = \beta \rho_-$. Then, $||f||_2^2 \leq C \sum_{i=0}^{\infty} (d-1)\beta r^i = C|\beta|^2 \sum_{i=0}^{\infty} ((d-1)r^2)^i$. Since $r < \frac{1}{\sqrt{d-1}}$, we obtain $(d-1)r^2 < 1$ and, thus, $||f||_2^2 < \infty$. Hence, $f \in l_2$. It is clear that it satisfies (2) for all $i \geq 1$. We just have to check it satisfies $\lambda f_0 = df_1 + \lambda$, i.e., $\lambda \beta = d\beta - \lambda + 1$. One can check that this holds for some choice of $\beta$. So, if $\lambda > 2\sqrt{d-1}$, then $\lambda \in \sigma(A_T)$.

A similar argument shows that if $\lambda < -2\sqrt{d-1}$, then $\lambda \in \sigma(A_T)$.

3 A proof of the Alon-Boppana lower bound

We proceed now to the proof of theorem 1.1. In this section, $G$ is a graph as in theorem 1.1 and $A = A_G$. It is not hard to see that $\lambda_2(G) = \max_{f \neq 1} \frac{f^T A f}{||f||}$ (where $1$ denotes the constant function which maps everything to 1). So, we will define a convenient $f$ that will give us the required lower bound for $\lambda_2(G)$.

Strategy of the proof: Consider $\Delta = \text{diam}(G)$ and $s,t \in V(G)$ such that $\text{dist}(s,t) = \Delta$. Roughly speaking, we will define $f$ such that its values for vertices ‘near’ $s$ are positive, its values for vertices ‘near’ $t$ are negative and the remaining ones are mapped to zero. More specifically, we let $k = \lfloor \Delta^2 \rfloor - 1$ and consider $T_{d,k}$, the $d$-‘regular’ tree of height $k$ (see figure 1). We construct an eigenvector $g$ for $A_{T_{d,k}}$ (the adjacency matrix of $T_{d,k}$) whose eigenvalue satisfies $\mu \geq 2\sqrt{d-1}(1 - \frac{1}{d^2})$. By defining the values of $f$ according to the values of $g$ in a certain way (and normalizing its positive and negative values such that $<f,1> = \sum f(x) = 0$), we can show that $\frac{f^T A f}{||f||^2} \geq \mu$, giving us the lower bound we wanted.

We want to construct an eigenvector $g$ for $A_{T_{d,k}}$ (with eigenvalue $\mu$). If we assume $g$ is spherical around $v$ (the root of $T_{d,k}$), we get the following equations for $g$:

$$
\begin{align*}
\mu g_0 &= dg_1 \\
\mu g_i &= g_{i-1} + (d-1)g_{i+1}, \text{ for } i = 1, \ldots, k \\
g_{k+1} &= 0
\end{align*}
$$

(to simplify notation we assume there is a $(k+1)$-th level and the value
Claim 3.1. There is a $\mu > 1 - \frac{c}{\Delta^2}$ (with $c \approx 2\pi^2$) such that there is a real solution $g$ of (3) that is non-negative and non-increasing.

Proof. Define $h : \{0, \ldots, k+1\} \to \mathbb{R}$ by $h(i) := (d-1)^{-\frac{i}{2}} \sin((k+1-i)\theta)$. It is easy to see that $h_{k+1} = 0$. Let us check that $h$ satisfies (3) regardless of the value of $\theta$:

$$h_{i-1} + (d-1)h_{i+1} = (d-1)^{-\frac{i+1}{2}} \cdot [\sin((k+2-i)\theta) + \sin((k-i)\theta)]$$

$$= \sqrt{d-1}(d-1)^{-\frac{i}{2}} \cdot 2 \sin((k+1)\theta) \cos(\theta) = \mu h_i$$

The condition for $i = 0$ reads

$$(2d - 2) \cos(\theta) \sin((k+1)\theta) = d \sin(k\theta)$$

The smallest positive root of this equation is in $(0, \frac{\pi}{k+1})$ because the difference of the two terms of this equation change sign between $0$ and $\frac{\pi}{k+1}$. So, $\theta \in (0, \frac{\pi}{k+1})$. Hence, $\theta_0 < \frac{\pi}{k+1} \approx \frac{2\pi}{k}$, since $k = \lfloor \frac{\Delta}{2} \rfloor - 1$. By the Taylor expansion of cos, $\cos(\theta_0) > 1 - \frac{c}{\Delta^2}$ (so $c \approx 2\pi^2$).

Moreover, since $\theta \in (0, \frac{\pi}{k+1})$, $h$ is non-negative and non-decreasing.\qed

Let $s$ and $t$ be two vertices that realize the distance $\Delta$. We define the sets of points ‘near’ $s$, ‘near’ $t$ and the rest of them:

$$S_i := \{v : \text{dist}(s, v) = i\} \quad \text{for } i = 0, \ldots, k$$

$$T_i := \{v : \text{dist}(t, v) = i\} \quad \text{for } j = 0, \ldots, k$$

$$Q := V(G) \setminus \bigcup_{0 \leq i \leq k} (S_i \cup T_i)$$

Notice that the sets $S_i$ and $T_j$ are disjoint (for any $i, j$). We are now ready to define $f : V(G) \to \mathbb{R}:

$$f(v) = \begin{cases} 
    c_1g_i & \text{if } v \in S_i, \\
    -c_2g_i & \text{if } v \in T_i \\
    0 & \text{otherwise}
\end{cases}$$

where $c_1$ and $c_2$ are positive constants that will be determined later.

Claim 3.2. With this definition we have $$(Af)_v \geq \mu f_v \quad \text{for } v \in \cup_i S_i$$

and $$(Af)_v \leq \mu f_v \quad \text{for } v \in \cup_i T_i$$
Proof. Let $v \in T_i$ for some $i > 0$. Then, of its neighbors, $p \geq 1$ belong to $T_{i-1}$, $q$ belong to $T_i$ and $(d - p - q)$ belong to $T_{i+1}$. Thus,

$$(Af)_v = -(p \cdot c_2 g_{i-1} + q \cdot c_2 g_i + (d - p - q) \cdot c_2 g_{i+1})$$

Now, by (3) and claim 3.1,

$$(Af)_v = -c_2 \cdot (pg_{i-1} + qg_i + (d - p - q)g_{i+1})$$

$$\leq -c_2 \cdot (g_{i-1} + (p - 1)g_{i-1} + qg_i + (d - p - q)g_{i+1})$$

$$= -c_2 \cdot (g_{i-1} + (d - 1)g_{i+1})$$

$$= -c_2 \cdot (A_{d,k} g)_i = -c_2 \mu g_i = \mu f_v.$$

A similar argument works for $v \in S_i$. \hfill \Box

As a consequence of claims 3.1 and 3.2, we obtain the following

**Theorem 3.3 (Alon-Boppana).**

$$\lambda_2(G) \geq 2\sqrt{d - 1} \left(1 - \frac{c}{\Delta^2}\right)$$

**Proof.** By claim 3.2,

$$f^T A f = \sum_{v \in V(G)} f_v(Af)_v$$

$$= \sum_{v \in U S_i} f_v(Af)_v + \sum_{v \in U T_i} f_v(Af)_v + \sum_{v \in Q} f_v(Af)_v$$

$$\geq \sum_{v \in U S_i} f_v \mu f_v + \sum_{v \in U T_i} f_v \mu f_v = \mu f^T f = \mu \|f\|^2$$

Finally, by choosing suitable $c_1$ and $c_2$, we get

$$\sum_{v \in U S_i} f_v = - \sum_{v \in U T_i} f_v$$

and, thus, $f \perp 1$.

Therefore, by claim 3.1,

$$\lambda_2(A) \geq \frac{f^T A f}{\|f\|^2} \geq \mu \geq 2\sqrt{d - 1} \left(1 - \frac{c}{\Delta^2}\right)$$

\hfill \Box

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4 Further Remarks

**Conjecture 4.1.** For every integer \(d \geq 3\) there exists arbitrarily large \(d\)-regular Ramanujan graphs.

**Theorem 4.2** (Lubotzky-Phillips-Sarnak [LPS88], Margulis [Mar88], Morgenstern [Mor94]). For every prime \(p\) and every positive integer \(k\) there exist infinitely many \(d\)-regular Ramanujan graphs with \(d = p^k + 1\).

References


