# MAT667 Expander Graphs: Lecture 1

These notes have been designed to introduce the uninitiated reader to the basic theory of graphs. Much of it follows the first chapter of *Elementary Number Theory, Group Theory, and Ramanujan Graphs* by Davidoff, Sarnak and Valette, which the author found to be a very clear recounting of basic graph theory definitions and which also contains a number of excellent problems at the end of each section. We also use the main text for this course, *Expander Graphs and their Applications* by Hoory, Linial and Wigderson.

**Basic Definitions**

We begin by setting up notation. Define a graph $X$ to be the pair of sets $(V, E)$, where $V$ is a collection of vertices and $E$ a collection of edges. If both $|V|$ and $|E|$ are finite, we shall say $X$ is finite. Note that *loops* (edges connecting a vertex to itself) are allowed, as are multiple edges connecting the same pair of vertices. We say that a graph is *oriented* or *directed* if we choose to denote an originating vertex and a terminating vertex for each edge. In such a case, given edge $e$ we shall let $e^-$ and $e^+$ represent the corresponding origin and terminal vertices. Unless stated otherwise, assume that any graph is undirected. If two vertices are connected by an edge, we shall say they are *adjacent* or *neighbouring.*

For any arbitrary graph $X$, let

$$l^2(V) := \{f : V \to \mathbb{C} \mid \sum_{v \in V} |f(v)|^2 < \infty\}$$

$$\text{and } l^2(E) := \{f : E \to \mathbb{C} \mid \sum_{e \in E} |f(v)|^2 < \infty\}$$

Obviously in the case where $|V| = n$ is finite, $l^2(V)$ reduces to $\mathbb{C}^n$ as every such function can be thought of as an $n \times 1$ vector and likewise the same holds for $l^2(E)$ when $E$ is a finite set.

**Definition 1.** *Let $X = (V, E)$ and $X' = (V', E')$ be two graphs. We say that $X$ and $X'$ are* **isomorphic** *if there exists a bijection $f : V \to V'$ such that for all pairs of vertices $u, v \in V$, the number of edges connecting $u$ and $v$ equals the the number of edges connecting $f(u)$ and $f(v)$. In such a case $f$ is called an* **edge-preserving isomorphism**.

We can of course extend the definition of isomorphism to directed graphs as well, by placing a natural restriction on the orientation of edges connecting $f(u)$ and $f(v)$. We shall not however have reason to use this.

**Definition 2.** *Let $X$ be an $n$-graph (ie. a graph with $n = |V|$ vertices) and let $v_1, \ldots, v_n$ denote some arbitrary fixed ordering of the vertices. The* **adjacency matrix** *of $X$, denoted $A = A(X)$ is the $n \times n$ matrix $(a_{ij})$ where $a_{ij}$ equals the number of edges connecting $v_i$ and $v_j$.*

The adjacency matrix will be our main focus of study in this first lecture. It is clear from the definitions that two undirected graphs are isomorphic iff they have the same adjacency matrix, possibly after reordering vertices. Hence it is natural to say that the adjacency matrix completely determines the graph and vice-versa and thus one would hope that we could recover interesting graph theoretic properties by studying the algebraic properties of $A$. This is precisely what we shall do for the remainder of the lecture.

Perhaps the most obvious feature of $A(X)$ is that it is always real and symmetric (and of course every entry is positive). It follows from the Spectral Theorem of linear algebra that there exists an orthonormal basis of $\mathbb{R}^n$ consisting of eigenvalues of $A$. This particular property will play a pivotal role in what is to follow and we shall return to it after a few more definitions.

**Definition 3.** *Let $v$ be some vertex of a graph $X$. Define the* **degree** *of $v$ ( $deg(v)$ ) to be the number of edges incident to $v$. We say that a graph $X$ is* **k-regular** *if the degree of every vertex is $k$.*

**Remark** In the definition of degree, there is the issue concerning whether to count a loop twice. Following what appears to be the convention in HLW, we have decided not to do this.

Notice that these two concepts are completely encoded in the rows of the adjacency matrix for any finite graph. Indeed if $v = v_i$, then $\deg(v) = \sum_j a_{ij}$ and so a finite graph is $k$-regular iff this sum is $k$ for all appropriate $i$.

**Definition 4.** *We say that a graph $X$ is* **simple** *if there are no multiple edges.*

Again if $X$ is finite, the simplicity of $X$ is encoded by the fact that $a_{ij} \in \{0, 1\}$.

**Definition 5.** *A* **path** *is a sequence of vertices in a graph $X$ such that for each consecutive pair $(v_i, v_{i+1})$, there is an edge connecting the two. If there exists a finite path whose initial vertex is $v$ and whose final vertex is $v'$, we say that $v$ and $v'$ are* **connected**. *A graph is* **connected** *if every pair of vertices are.*

It is intuitively clear that being "connected" forms an equivalence relation (note that the sequence $(v)$ is a path connecting $v$ with itself) and that this partitions vertices into disjoint "connected components". A graph is thus connected iff every vertex belongs to the same component.

We shall now focus our attention on graphs that are both finite and regular. Henceforth unless otherwise noted, we shall assume $X$ denotes a finite, $d$-regular graph with $n$ vertices. In such a case, we shall see that the property of connectedness is also easily recoverable from the adjacency matrix $A$ of $X$.

As noted earlier, $A$ diagonalizes over the reals, so let $\lambda_0 \geq \lambda_1 \geq \ldots \lambda_{n-1}$ denote the $n$ real eigenvalues of $A$. We shall call these eigenvalues the *spectrum* of $X$. Because the entries of $A$ are so closely connected with pairings of the $n$ vertices of $X$, it is often convenient in what follows to index these entries with respect to vertices instead of $i, j$. Hence we write

$$A = (a_{ij}) = (a_{xy})_{x,y \in V}.$$

Consider $A$ as a linear transformation on the space $l^2(V)$, and one gets the following:

$$\begin{aligned} Af &= \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{pmatrix} \begin{pmatrix} f(v_1) \\ f(v_2) \\ \vdots \\ f(v_n) \end{pmatrix} \\ &= \begin{pmatrix} a_{11}f(v_1) + a_{12}f(v_2) + \ldots + a_{1n}f(n) \\ a_{21}f(v_1) + a_{22}f(v_2) + \ldots + a_{2n}f(n) \\ \vdots \\ a_{n1}f(v_1) + a_{n2}f(v_2) + \ldots + a_{nn}f(n) \end{pmatrix}. \end{aligned}$$

Thus indexing by vertices we have the formula $Af(v) = \sum_{x \in V} a_{vx} f(x)$.

**Proposition 6.** *Let $X$ be as above. Then*
*(a) $\lambda_0 = d$;*
*(b) $|\lambda_i| \leq d$ for $1 \leq i \leq n-1$;*
*(c) $\lambda_0$ has multiplicity 1 iff $X$ is connected.*

*Proof.* By the remarks following the definition of regularity, we see that the constant function $(1, 1, \ldots, 1)$ is an eigenfunction of $A$ with eigenvalue $d$. We next show that any other eigenvalue has norm less then $d$. Indeed let $f$ be some arbitrary eigenfunction on $V$ with eigenvalue $\lambda$ and suppose $x \in V$ is such that $|f(x)| \geq |f(y)|$ for all $y \in V$ (recall of course that $V$ is finite). Then we have

$$\begin{aligned} |\lambda||f(x)| &= |Af(x)| \\ &= |\sum_{y \in V} a_{xy} f(y)| \\ &\leq \sum_{y \in V} |a_{xy}||f(x)| \\ &\leq |f(x)| \sum_{y \in V} a_{xy} \qquad \text{because the matrix entries are positive} \\ &= d \cdot |f(x)| \qquad \text{which completes the proof of both } (a) \text{ and } (b) \end{aligned}$$

For (c), assume first $X$ is connected and suppose $f$ is an eigenfunction with eigenvalue $\lambda_0 = d$. Again let $x$ be the vertex where $f$ achieves its maximum absolute value, and WLOG we may assume $f$ is real-valued with $f(x) > 0$ (due to the spectral theorem). Then we have $f(x) = \sum_{y \in V} \frac{a_{xy}}{d} f(y)$. Note that $\sum_{y \in V} \frac{a_{xy}}{d}$ is a convex sum of positive numbers and since $f(x) \geq f(y)$, it follows that $f(y) = f(x)$ whenever $a_{xy} \neq 0$ or equivalently $f(y) = f(x)$ if $y$ is adjacent to $x$. It is clear that we can repeat this argument by replacing $x$ with any adjacent vertex to conclude that $f$ is constant on any vertex which is within two edges of $x$. Inductively, we get that $f$ must be constant on any vertex that is connected to $x$ and since $X$ is connected we conclude $f$ must be constant and hence $\lambda_0$ has multiplicity 1.

Now assume $X$ is not connected. Let $V_1$ be a connected component of $V$ and let $E_1$ denote all the edges connecting vertices of $V_1$. It is clear that we can then partition $X$ into two disjoint subgraphs $X_1 = (V_1, E_1)$ and $X_2 = (V \backslash V_1, E \backslash E_1)$.

Furthermore, if we ensure that the vertices of each graph are labelled in the appropriate manner, the adjacency matrix of $X$ will look like

$$A(X) = \begin{pmatrix} A(X_1) & 0 \\ 0 & A(X_2) \end{pmatrix}.$$

Basic linear algebra tells us that the eigenvalues of $A(X)$ must then be the same as those for either of the two submatrices and the multiplicity for any value will be the sum of the multiplicities of that value for $A(X_1)$ and $A(X_2)$. By $(a)$, $d$ is the largest eigenvalue for both $A(X_1)$ and $A(X_2)$ which completes the proof. $\square$

Before moving on to the concept of expander graphs, we introduce one more basic graph theoretic notion that is again encapsulated by the study of eigenvalues on the adjacency matrix.

**Definition 7.** *An arbitrary graph $X$ (need not be finite or connected or regular) is* **bipartite** *if there exists a partition of vertices $V = V_1 \cup V_2$ such that for any two vertices $x, y \in V$, if $a_{xy} \neq 0$ then either $x \in V_1$ and $y \in V_2$ or vice-versa.*

From a conceptual point of view, being bipartite amounts to being able to colour the vertices of a graph using two colours in such a manner as to ensure no two adjacent vertices have the same colour. The following proposition shows that begin bipartite is reflected in the symmetry of the spectrum of a graph.

**Proposition 8.** *Let $X$ be as usual. The following are equivalent:*
*(i) $X$ is bipartite;*
*(ii) The spectrum of $X$ is symmetric about 0;*
*(iii) $\lambda_{n-1} = -d$.*

*Proof.* We show $(i) \Rightarrow (ii) \Rightarrow (iii) \Rightarrow (i)$.

$(i) \Rightarrow (ii)$: Writing $V = V_1 \cup V_2$ as in the definition of bipartite, suppose $f$ is an eigenfunction of $A$ with eigenvalue $\lambda$. Define the function $g$ on $V$ as the following:

$$g(x) := \begin{cases} f(x) & \text{if } x \in V_1 \\ -f(x) & \text{if } x \in V_2. \end{cases}$$

Then

$$\begin{aligned} Ag(x) &= \sum_{y \in V} a_{xy} g(y) \\ &= \sum_{y \in V_1} a_{xy} f(y) - \sum_{y \in V_2} a_{xy} f(y). \end{aligned}$$

Since $X$ is bipartite, every term in one of the two sums above will vanish, depending upon whether $x$ is in $V_1$ or $V_2$. If say $x$ is in $V_1$ then the first sum vanishes and hence $Ag(x) = -\sum_{y \in V_2} a_{xy} f(y) = -\sum_{y \in V} a_{xy} f(y) = -\lambda f(x)$. The case when $x$ is in $V_2$ is similar.

$(ii) \Rightarrow (iii)$: This follows immediately from part $(a)$ of the previous proposition.

$(iii) \Rightarrow (i)$: Let $f$ be a real function that corresponds to eigenvalue $-d$. Let $x$ be a vertex where the maximum absolute value of $f$ is attained. We will show that $f$ attains only two values, namely $f(x)$ and $-f(x)$, and partitioning the vertices to correspond with their values under $f$ will realize $X$ as bipartite. The proof will be similar to part $(c)$ of the previous proposition. Note first that we have

$$-d \cdot (f(x)) = Af(x) = \sum_{y \in V} a_{xy} f(y)$$

and since the $a_{xy}$ are all positive and sum to d, it follows by the maximality in modulus of $f(x)$ that $f(y) = -f(x)$ whenever $a_{xy} \neq 0$. Hence $f(y) = -f(x)$ for all $y$ adjacent to $x$. But now as before we can repeat the argument with any such $y$ replacing $x$. Since the graph is connected, it follows that this argument will inductively show that the image of $f$ lies in $\{\pm f(x)\}$ and that $y$ is adjacent to $z$ only if $f(y) \neq f(z)$ for any two arbitrary vertices $y, z \in V$. This completes the proof. □

**Expander Graphs and Inequalities**

We now delve a little deeper into graph theory and introduce more sophisticated machinery that will be the main focus of the course. As before, we will always assume the $X$ is a connected, undirected, $d$-regular $n$-graph unless noted otherwise. For $S, T \subseteq V$ let the set $E(S,T)$ denote the subset of edges connecting $S$ with $T$. More explicitly, we have

$$E(S,T) = \{(u,v) \mid u \in S, v \in T, (u,v) \in E\}.$$

If we consider the set of undirected edges of $E$ as pairs of directed edges, then $E(S,T)$ can be thought of as a set of directed edges. We also denote $E(S) = E_S$ as the collection of edges that connect vertices both lying in $S$.

**Definition 9.** *The* **edge boundary** *of a subset of vertices $S$, denoted $\partial S$, is $E(S, \overline{S})$. We define the* **edge expansion ratio** *(or* **isoperimetric constant** *or* **expanding constant***) of a graph $X$, denoted $h(X)$, as:*

$$h(X) = \min_{\{S \mid |S| \leq \frac{n}{2}\}} \frac{|\partial S|}{|S|}.$$

As noted in the introductory lecture, we could equally well define a **vertex boundary**, which counts the number of vertices adjacent to some subset and also apply a type of expanding constant to this value. We shall not however have reason to do this in the current lecture.

**Definition 10.** *Let $(X_m)$ be a sequence of $d$-regular graphs with size (in terms of number of vertices) increasing with $m$. We say that this is a sequence of* **expander graphs** *if there exists some constant $\epsilon > 0$ such that $h(X_m) \geq \epsilon$ for all $m$.*

**Examples of Expander Graphs**

(1) A family of 8-regular graphs $X_m$ for $m \in \mathbb{Z}_{>0}$. Denote the vertex set $V_m = \mathbb{Z}/m\mathbb{Z} \times \mathbb{Z}/m\mathbb{Z}$. For vertex $(x,y)$ the neighbours are $\{(x+y,y), (x-y,y), (x,y+x), (x,y-x), (x+y+1,y), (x-y+1,y), (x,y+x+1), (x,y-x+1)\}$ (where addition is modulo $m$).

This was the first explicitly constructed family of expander graphs and is due to Margulis using representation theory. Gabber and Galil later provided a bound on the expansion ratio using harmonic analysis. A proof of expansion can be found in section 8 of HLW.

(2) A family of $p$-regular $p$-graphs for every prime $p$. Here $V_p = \mathbb{Z}_p$ and the neighbours of a point $x$ are $\{x+1, x-1, x^{-1}\}$ where all operations are mod $p$ and the inverse of 0 is defined as 0. Proof of expansion relies on the Selberg 3/16 theorem and is discussed in section 11 of HLW.

We now prepare ourselves to prove two basic inequalities of expander graph theory. Before we can proceed to do so however, we are going to develop a little more machinery that exploits the $l^2$ spaces introduced earlier. Let $X$ be as usual, and endow $X$ with an arbitrary orientation. We define the *simplical coboundary operator* $\delta : l^2(V) \to l^2(E)$ as:

$$(\delta f)(e) = f(e^+) - f(e^-).$$

As a matrix, $\delta$ can be thought of as the incidence matrix of $X$ with the endowed orientation. More explicitly, $[\delta]$ is an $|E| \times |V|$ matrix whose coordinates $\delta_{e,v}$ are equal to:

$$\delta_{e,v} = \begin{cases} 1 & \text{if v is the head of e} \\ -1 & \text{if v is the tail of e} \\ 0 & \text{if e is a loop at v or otherwise.} \end{cases}$$

Notice that $l^2(V)$ has a natural scalar product, $< f, g >= \sum_{v \in V} \overline{f(v)} g(v)$ and likewise for $l^(E)$. Hence $\delta$ is a linear operator between two finite dimensional $\mathbb{C}$ vector spaces and thus it has an adjoint $\delta^* : l^2(E) \to l^2(V)$ whose action is defined by $< \delta f, g >=< f, \delta^* g >$ for $f \in l^2(V), g \in l^2(E)$. As a matrix, obviously $[\delta^*]$ is just the conjugate of $[\delta]$ (note of course that $[\delta]$ is real). To explicitly compute the value of $\delta^* g$ at vertex $v$, take $f \in l^2(V)$ to be the function that is 1 at $v$ and 0 else. Then

$$\begin{aligned} \delta^* g(v) &= < f, \delta^* g > \\ &= < \delta f, g > \\ &= \sum_{e \in E_1} g(e) - \sum_{e \in E_2} g(e) \end{aligned}$$

where $E_1$ and $E_2$ are the sets of edges ending or originating from $v$ respectively.

Obviously $\delta$ and its conjugate depend explicitly on the orientation chosen for the graph $X$. However, the remarkable fact is that their composition $\delta^*\delta : l^2(V) \to l^2(E)$ does not. Indeed we have

$$\begin{aligned} (\delta^*\delta f)(v) &= \sum_{e \in E_1} \delta f(e) - \sum_{e \in E_2} \delta f(e) \\ &= \sum_{e \in E_1} (f(v) - f(e^-)) - \sum_{e \in E_2} (f(e^+) - f(v)) \\ &= d \cdot f(v) - \sum_{(v,u) \in E} f(u) \\ &= d \cdot f(v) - Af(v). \end{aligned}$$

Hence the operator $\delta^*\delta$ depends only on the adjacency matrix of $X$ and can be written succinctly as $\delta^*\delta = d \cdot Id - A$. This operator is known as the (minus) *combinatorial laplacian* for $X$ and the operator $\delta$ can be thought of as the gradient for $X$ endowed with the given orientation. Notice that the eigenvalues of $\delta^*\delta$ are $0 \leq d-\lambda_1 \leq d-\lambda_2 \leq \ldots \leq d-\lambda_{n-1}$. It follows that if $f \in l^2(V)$ is orthogonal to the constant functions, then $||\delta f||^2 = |<\delta^*\delta f, f>| \geq (d-\lambda_1)\cdot||f||^2$. The value $(d-\lambda_1)$ is going to be very important in studying certain expansion properties of a graph $X$ and is called the *spectral gap* of $X$.

**Theorem 11.** *(Alon-Milman Inequalities) Let $X$ be a connected, undirected, $d$-regular $n$-graph with spectrum $d=\lambda_0 \geq \lambda_1 \geq \ldots \geq \lambda_{n-1}$. Then*

$$\frac{d-\lambda_1}{2} \leq h(X) \leq \sqrt{2d(d-\lambda_1)}.$$

*Proof.* We will follow Alon and Milman's paper $\lambda_1$*, Isoperimetric Inequalities for Graphs, and Superconcentrators* for the first inequality, the so called "easy half". In fact we will prove a more general inequality. Let $A$ and $B$ be two disjoint subsets of $V$ and let $\rho$ be the distance between them (distance is defined as the minimum number of edges needed to construct a path between 2 vertices in $A$ and $B$). Let $E_A$ and $E_B$ be the set of edges whose vertices are both in $A$ or both in $B$ respectively. We claim that it suffices to show the following inequality:

$$d-\lambda_1 \leq \frac{1}{\rho^2}\left(\frac{1}{|A|}+\frac{1}{|B|}\right)(|E|-|E_A|-|E_B|). \tag{1}$$

Indeed, suppose the above held. Then take $A$ to be any subset of vertices such that $|A| \leq |V|/2$ and take $B = V\backslash A$. Then $|E|-|E_A|-|E_B| = \partial A$, $\rho = 1$ and $(\frac{1}{|A|}+\frac{1}{|B|}) \leq \frac{2}{|A|}$. Rearranging would then give the desired result. Hence we proceed to prove (1).

Define $g \in l^2(V)$ by

$$g(v) = \frac{n}{|A|} - \frac{1}{\rho}\left(\frac{n}{|A|}+\frac{n}{|B|}\right)\min(\rho(v,A),\rho).$$

where $\rho(v,A)$ is the distance as defined above between $\{v\}$ and $A$. Note that if $v \in A$ then the distance is 0, so $g(v) = \frac{n}{|A|}$ and if $v \in B$, $g(v)$ reduces to $-\frac{n}{|B|}$. If $u, v$ are neighbours, $\rho(u,A)$ and $\rho(v,A)$ cannot differ by more then one, and hence $|g(u)-g(v)| \leq n/\rho(1/|A|+1/|B|)$. It follows immediately from the definitions that

$$|\delta g(e)| \leq n/\rho(1/|A|+1/|B|) \qquad \text{for all } e \in E.$$

Now define the constant $\alpha = 1/n \sum_{v\in V} g(v)$ and let $f = g-\alpha$. Then $f$ is orthogonal to the constant functions since $\sum_v f(v) = 0$. Hence by the earlier remark we have

$$\begin{aligned}
(d-\lambda_1)\left(\frac{n^2}{|A|}+\frac{n^2}{|B|}\right) &\leq (d-\lambda_1)\left(\left(\frac{n}{|A|}-\alpha\right)^2\cdot|A|+\left(\frac{n}{|B|}+\alpha\right)^2\cdot|B|\right)\\
&= (d-\lambda_1)\sum_{v\in A\cup B} f(v)^2\\
&\leq (d-\lambda_1)||f||^2\\
&\leq ||\delta f||^2\\
&= \sum_{e\in E}(f(e^+)-f(e^-))^2 \qquad \text{for some arbitrary orientation of } X\\
&= \sum_{e\in E}(g(e^+)-g(e^-))^2\\
&= \sum_{e\in E\setminus(E_A\cup E_B)}(g(e^+)-g(e^-))^2\\
&\leq \frac{1}{\rho^2}\left(\frac{n}{|A|}+\frac{n}{|B|}\right)^2(|E|-|E_A|-|E_B|).
\end{aligned}$$

Dividing both sides by $\left(\frac{n^2}{A}+\frac{n^2}{B}\right)$ yields the desired result and proves the first half of the Alon-Milmon inequalities.

The second inequality, $h(X) \leq \sqrt{2d(d-\lambda_1)}$ is somewhat more involved. The proof follows that found in DSV starting on page 14, although some details are omitted. We proceed by inducing some arbitrary orientation on $X$ and then prove 3 inequalities involving the constant

$$B_f = \sum_{e\in E}|f(e^+)^2 - f(e^-)^2|$$

when $f$ is a real valued non-negative function in $l^2(V)$ (clearly $B_f$ does not depend on the orientation). Denote by $\beta_r > \beta_{r-1} > \ldots > \beta_0 \geq 0$ the distinct values attained by such an $f$, and define $L_i = \{x \in V : f(x) \geq \beta_i\}$. Notice that $\partial L_i$ consists of edges where $f$ attains a value greater then $\beta_i$ and a value strictly less then $\beta_i$ on the two defining vertices.

**Step 1:** $B_f = \sum_{i=1}^r |\partial L_i|(\beta_i^2 - \beta_{i-1}^2)$.

To see this equality, first reduce the formula for $B_f$ to summing over edges for which $f(e^+) \neq f(e^-)$. Denote all such edges $E_f$. For any edge $e \in E_f$, let $i(e) > j(e)$ denote the indices in $\{0,\ldots,r\}$ which correspond to the two distinct values $f$ attains on $e^+$ and $e^-$. Then we have

$$\begin{aligned}
B_f &= \sum_{e\in E_f}(\beta_{i(e)}^2 - \beta_{j(e)}^2)\\
&= \sum_{e\in E_f}\sum_{l=j(e)+1}^{i(e)}(\beta_{l(e)}^2 - \beta_{l(e)-1}^2).
\end{aligned}$$

All that remains now is to count how many times the term $(\beta_i^2 - \beta_{i-1}^2)$ appears in the final sum above for each $i$. But notice that such a term will appear for a given edge $e$ iff $f$ attains a value $\geq \beta_i$ on one vertex and a value $< \beta_i$ on the other, aka. if $e \in \partial L$.

**Step 2:** $B_f \leq \sqrt{2d}||\delta f|| \cdot ||f||$.

Factoring $B_f$ and using Cauchy-Schwartz and the $AM-GM$ inequality, we get:

$$\begin{aligned}
B_f &= \sum_{e \in E} |f(e^+) - f(e^-)| \cdot |f(e^+) + f(e^-)| \\
&\leq \left(\sum_{e \in E} (f(e^+) - f(e^-))^2\right)^{1/2} \left(\sum_{e \in E} (f(e^+) + f(e^-))^2\right)^{1/2} \\
&\leq \sqrt{2} \left(\sum_{e \in E} f(e^+)^2 + f(e^-)^2\right)^{1/2} ||\delta f|| \\
&= \sqrt{2d} \left(\sum_{v \in V} f(v)^2\right)^{1/2} ||\delta f|| \\
&= \sqrt{2d}||\delta f|| \cdot ||f||.
\end{aligned}$$

**Step 3:** Assume $|\text{supp } f| \leq \frac{|V|}{2}$. Then $B_f \geq h(X)||f||^2$.

Note that $\beta_0 = 0$ and $|L_i| \leq |V|/2$ for all $i = 1, 2, \ldots, r$. Hence by definition, $|\partial L_i| \geq h(X)|L_i|$ for all such $i$. Using this in conjunction with step 1 we get

$$\begin{aligned}
B_f &\geq h(X) \sum_{i=1}^{r} |L_i|(\beta_i^2 - \beta_{i-1}^2) \\
&= h(X)(\beta_r^2|L_r| + \beta_{r-1}^2(|L_{r-1}| - |L_r|) + \ldots + \beta_1^2(|L_1| - |L_0|)).
\end{aligned}$$

But notice that $|L_{i-1}| - |L_i|$ is precisely the number of vertices at which $f$ attains the value $\beta_i$. The result follows.

We now use these results on a carefully selected function. Let $g$ be a real-valued eigenfunction of $\delta^*\delta$ with respect to $d - \lambda_1$. By multiplying by negative 1 if necessary, we may assume that $g$ is positive on no more then half of the vertices (Note that $g$ must be positive on at least one vertex because it is non-zero and orthogonal to the constant functions). Define $f$ as $f(v) = \max\{g(v), 0\}$ so that $f$ satisfies the support requirements of step 3. Let $V^+$ denote the support of $f$. Note that for $v \in V^+$ we have

$$\begin{aligned}
\delta^*\delta f(v) &= d \cdot f(v) - \sum_{u \in V} a_{vu} f(u) \\
&= d \cdot g(v) - \sum_{u \in V^+} a_{vu} g(u) \\
&\leq d \cdot g(v) - \sum_{u \in V} a_{vu} g(u) \\
&= \delta^*\delta g(v) = (d - \lambda_1) g(v).
\end{aligned}$$

It follows that we can put a bound on the norm of $\delta f$ since we have

$$\begin{aligned} ||\delta f||^2 &= <\delta^* \delta f, f>= \sum_{v \in V^+} (\delta^* \delta f(v)) g(v) \\ &\leq (d-\lambda_1) \sum_{v \in V^+} g(v)^2 = (d-\lambda_1)||f||^2. \end{aligned}$$

Combining this with the results from step 2 and 3 we get

$$h(X)||f||^2 \leq B_f \leq \sqrt{2d}||f|| \cdot ||\delta f|| \leq \sqrt{2d(d-\lambda_1)}||f||^2.$$

□

**Corollary 12.** *A sequence $\{X_m\}$ of finite, connected, d-regular graphs whose number of vertices increase with m is a family of expanders iff there exists some $\epsilon > 0$ such that the spectral gap for each graph is greater then $\epsilon$.*

**Lemma 13.** *(Expander Mixing Lemma) Given a d-regular graph with n vertices, let $\lambda = \lambda(G) = \max(|\lambda_1|, |\lambda_{n-1}|)$. Then for all $S, T \subseteq V$ :*

$$\left| |E(S,T)| - \frac{d|S||T|}{n} \right| \leq \lambda \sqrt{|S||T|}.$$

We remark that the quantity on the left hand side of the inequality is known as the *deviation* or *discrepancy*. On the one hand $|E(S,T)|$ measures the exact number of edges connecting $S$ and $T$, while the second term measures the average number of edges one would naively expect to connect two arbitrary sets of orders $|T|$ and $|S|$ in a $d$ regular graph with $n$ vertices.

*Proof.* Let $\{(1/\sqrt{n}, \ldots, 1/\sqrt{n}) = v_0, v_1, \ldots, v_{n-1}\}$ be an orthonormal basis of real eigenvectors wrt. the adjacency matrix, and let $1_S$ and $1_T$ be the characteristic functions of the sets $S$ and $T$ respectively. Note that by linearity we have the equality $< 1_S, A \cdot 1_T >= \sum_{x \in S} \sum_{y \in T} a_{xy} = |E(S,T)|$. Writing $1_S = \sum \alpha_i v_i$ and $1_T = \sum \beta_i v_i$ we have then

$$\begin{aligned} |E(S,T)| &= \sum_{i=0}^{n-1} \sum_{j=0}^{n-1} \alpha_i \beta_j < v_i, A \cdot v_j > \\ &= \sum_{i=0}^{n-1} \lambda_i \alpha_i \beta_i \qquad \text{by orthonormality} \\ &= \frac{d|T||S|}{n} + \sum_{i=1}^{n-1} \lambda_i \alpha_i \beta_i \end{aligned}$$

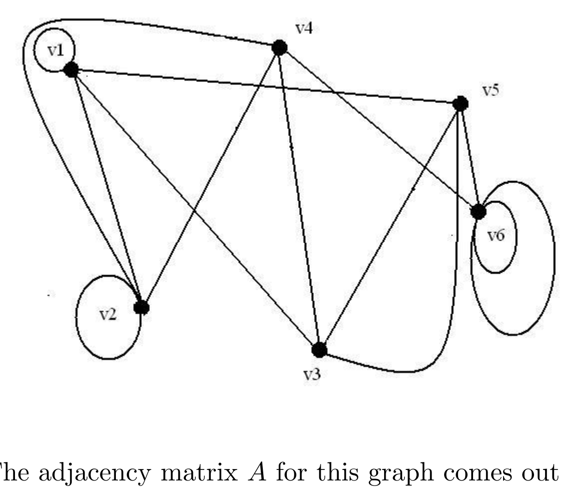
where the last equality uses the fact that $\alpha_0 =< 1_S, v_0 >= |S|/\sqrt{n}$ and likewise for $\beta_0$. Thus rearranging, we can rewrite the deviation as

$$\begin{aligned}\left||E(S,T)| - \frac{d|S||T|}{n}\right| &= \left|\sum_{i=1}^{n-1} \lambda_i \alpha_i \beta_i\right| \\ &\leq \sum_{i=1}^{n-1} |\lambda_i \alpha_i \beta_i| \\ &\leq \lambda \sum_{i=0}^{n-1} |\alpha_i \beta_i| \\ &\leq \lambda ||1_S|| \cdot ||1_T|| \qquad \text{by Cauchy-Schwartz.}\end{aligned}$$

Since the norms of $1_S$ and $1_T$ are easily seen to be $\sqrt{|S|}$ and $\sqrt{|T|}$, this completes the proof. $\square$

An interesting immediate result of the expander mixing lemma is the following. An *Independent set* of vertices $S \subseteq V$ is a collection of vertices such that $E(S,S)$ is empty. It follows from the EML that such a set can have size at most $\frac{n\lambda}{d}$ vertices. We say that the graph $X$ has a $k$-colouring if there exists a map $\psi : V \to \{1, \ldots, k\}$ such that $\psi(x) \neq \psi(y)$ if $x$ and $y$ are adjacent. The *chromatic number* of a graph is the smallest $k$ for which $X$ has a $k$-colouring. Since $\psi^{-1}(\{j\})$ must be an independent set for all $j$ and $\sum_j |\psi^{-1}(\{j\})| = n$, if $X$ has a $k$-colouring, then $k \geq \frac{d}{\lambda}$. In particular, the chromatic number of $X$ is less then or equal to $\frac{d}{\lambda}$.

We conclude the lecture with an example of some of the theory and language we have developed. Consider the graph $X$ shown below.




The adjacency matrix $A$ for this graph comes out as

$$A := \begin{pmatrix} 1 & 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 2 & 0 & 0 \\ 1 & 0 & 0 & 1 & 2 & 0 \\ 0 & 2 & 1 & 0 & 0 & 1 \\ 1 & 0 & 2 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 2 \end{pmatrix}.$$

It is easy to see both visually and from the matrix that this graph is 4-regular and is clearly not simple. Approximating the eigenvalues with Maple gives

$$\{4, 2.097, 2, 0.117, -1.509, -2.704\}$$

where the values 2 and 4 are exact. Hence the graph is not bipartite (which is in fact obvious since it contains loops) but is connected. The spectral gap is approximately 1.903, and thus the Alon-Milman inequalities puts bounds on the the expanding constant as $0.95 \leq h(X) \leq 2.76$. In fact if we consider the subset of vertices $S = \{v_1, v_3, v_5\}$, we find that $\partial S/|S| = 3/3 = 1$ and hence this must be $h(X)$ by considering possible denominators. Since $\lambda(X) \approx 2.704$, the expander mixing lemma tells us that the chromatic number of $X$ is at least $\lceil 4/2.704 \rceil = 2$, which isn't very useful. In fact, we know that 2 is not good enough because the graph is not bipartite. Colouring $v_1$ and $v_4$ green, $v_2$ and $v_5$ red, and $v_6$ and $v_3$ blue shows that $\chi(X) = 3$.