BUILDING OF $\mathrm{GL}_n(\mathbb{Q}_p)$

ANDREW FIORI

1. Introduction

So far in this seminar we have been focussed on graphs, in particular graphs that are good expanders. Here we will talk about a higher dimensional generalization of graphs along with the corresponding generalized concept of "being Ramanujan" for these objects. The generalization we shall be considering are "Simplicial Complexes" or more specifically buildings.

The primary references for these notes are:

- Lubotzky, Samuels and Vishne - Ramanujan Complexes of Type $\overline{A}_d$.
- Mark Ronan - Buildings: Main Ideas and Applications I and II.

2. Simplicial Complexes

Definition 2.1. A **Simplicial Complex** is the "gluing together" of simple building blocks called cells (or simplicies) in a manner which satisfies certain axioms.

- A 0-**cell** is a point.
- A 1-**cell** is a line connecting 2 points (0-cells).
- A 2-**cell** is a triangle filling in the area between 3 lines (1-cells).
- A 3-**cell** is a tetrahedron which is the volume within 4 triangles (2-cells).
- An n-**cell** is the connecting space for $(n+1)$ many $(n-1)$-cells.

The main axiom concerning gluing is that simplicies may only be glued along sub-simplicies.

One can say a lot of things about simplicial complexes in general but we shall stick with these simplistic notions for the purposes of these notes.

3. Construction of the Building for $\mathrm{GL}_n(\mathbb{Q}_p)$

We will now construct what is known as the "Affine Building" of $\mathrm{GL}_n(\mathbb{Q}_p)$. For now we will just say that a building is a special type of simplicial complex, we will talk briefly about what makes these special later.

Remark. One should note that most of the construction that follows works out in much the same way if we move to finite a extension of $\mathbb{Q}_p$ or to other algebraic groups.

We should also remark that the building for $\mathrm{GL}_n(\mathbb{Q}_p)$ is precisely the higher dimensional analog of the "Bruhat-Tits tree" for $\mathrm{GL}_2(\mathbb{Q}_p)$ which Atefeh discussed in the previous lecture.

Definition 3.1. A **(full)** $\mathbb{Z}_p$**-lattice** $L \subset \mathbb{Q}_p^n$ is a finitely generated free $\mathbb{Z}_p$ submodule such that $\mathbb{Q}_p \cdot L = \mathbb{Q}_p^n$.

We introduce an equivalence relation on the set of all lattices in $\mathbb{Q}_p^n$. We say two latices are equivalent, written $L_1 \sim L_2$, if there exists $C \in \mathbb{Q}_p$ such that $L_1 = CL_2$.

We will now construct the simplicial complex for the building X of $\mathrm{GL}_n(\mathbb{Q}_p)$, we do so by first defining the 0-cells, then the 1-cells, and so forth.

- The 0-cells (vertices) of X are the equivalence classes of lattices in $\mathbb{Q}_p^n$.

- The 1-cells (edges) of X are the lines which pairs $[L_1], [L_2]$ where we have $L_1 \supset L_2 \supset pL_1$ for some representatives L_1, L_2 of the equivalence classes.
- We take for d-cells every possible complete subgraph (in the already defined 1-skeleton) on $d+1$ many 0-cells.

  Equivalently, we can describe for example the 2 cells as all collections $[L_1], [L_2], [L_3]$ with $L_1 \supset L_2 \supset L_3 \supset pL_1$. This generalizes for d-cells.

We would like to be able to describe a bit the local structure of this complex, in particular to describe and classify the edges coming from a vertex. As we have seen for GL_2 one strategy is to look at the quotients of these lattices and work in a vector space over the residue field.

That is, for $L_0 \simeq \mathbb{Z}_p^n$ we have $L_0/pL_0 \simeq \mathbb{Z}_p^n/p\mathbb{Z}_p^n \simeq \mathbb{F}_p^n$. If we call this quotient map ϕ we see that if $L_0 \supset L_1 \supset pL_0$ then $\phi(L_1)$ is a subspace of $\mathbb{F}_p^n$. Conversely we remark that every subspace V of $\mathbb{F}_p^n$ corresponds to a different lattice $\phi^{-1}(V)$. We are consequently interested in describing the possible subspaces of $\mathbb{F}_p^n$. Moreover, we have under this correspondence that nested subspaces correspond to "Connected Vertices" and in particular we have that if $0 \subset V_1 \subset \cdots \subset V_r \subset \mathbb{F}_p^n$ then this corresponds to lattices $L_0 \supset L_r \supset L_{r-1} \supset \cdots \supset L_1 \supset pL_0$ which is precisely to say that an r-flag in $\mathbb{F}_p^n$ corresponds precisely to a $(r-1)$-cell containing L_0. From this it is clear that X is an $(n-1)$-simplicial complex as this is the length of a maximal flag.

Remark. We observe that the number of 1-dimensional subspaces in $\mathbb{F}_p^n$ is given by:

$$\frac{p^n - 1}{p - 1}$$

The number of 2-dimensional subspaces is:

$$\frac{(p^n - 1)(p^n - p)}{(p^2 - 1)(p^2 - p)}$$

The number of d-dimensional subspaces follow similar formulae.

In particular if we restrict to the X^1 (the 1-cells) the graph has a certain regularity.

4. Types of adjacency

We are interested in trying to give slightly more structure to the building. We observe that in the above counting of edges that a potentially interesting property of an oriented edge $L_1 \supset L_2 \supset pL_1$ is the dimension of the corresponding subspaces or equivalently the index of the one lattice in the other.

We define a map on the ordered 1-cells of X with values in $\mathbb{Z}/n\mathbb{Z}$ via:

$$\rho(L_1, L_2) \mapsto dim(L_1/L_2) = n - dim(\phi(L_2)).$$

This does depend on choice of ordering but only in that $\rho(L_1, L_2) = n - \rho(L_2, L_1)$.

We say the 0-cell L_2 is k**-adjacent** to L_1 if $\rho(L_1, L_2) = k$.

We saw in Luiz's talk that the usual "Adjacency Operator" acts as an operator on functions on vertices, we now extend this to the concept of colored adjacency. In particular we define:

$$L^2(X^0) = \left\{ f : X^0 \to \mathbb{C} | \quad \sum_{x \in X^0} |f(X)| < \infty \right\}$$

and we have the operator $A : L^2(X^0) \to L^2(X^0)$ given by:

$$Af(x) = \sum_{[x,y] \in X^1} f(y).$$

Analogously we define the operators $A_k : L^2(X^0) \to L^2(X^0)$ to be:

$$A_K(f)(x) = \sum_{\substack{[x,y]\in X^1 \\ \rho(x,y)=k}} f(y)$$

We remark that these operators are bounded (since the graph is regular), commuting (because 2-step paths of different sizes can be done in "either order"). Moreover the adjoint of A_k is A_{n-k} (which is based on the fact that$\rho(x,y) = n - \rho(y,x)$) and so the operators are normal.

We define the **Hecke Algebra** $H(\mathrm{GL}_n(\mathbb{Q}_p), \mathrm{GL}_n(\mathbb{Z}_p))$ to be the algebra generated by these operators.

Remark. Under the correct interpretation of modular forms as functions on lattices this is very similar to the usual Hecke Operator (at p).

One of the properties Luiz proved was a statement along the lines of saying that the non-trivial eigenvalues of the adjacency operator on a quotient of a regular tree have a relationship to the eigenvalues of the adjacency operator on the tree. Moreover, he defined a Ramanujan graph to be one where "The non-trivial eigenvalue of the adjacency operator is equal to an eigenvalue of the adjacency operator on its covering tree".

As such, we use this phrasing of being Ramanujan in order to generalize it to these higher dimensional complexes.

First, we define the simultaneous spectrum S of $(A_1, \ldots, A_n)$ to be:

$$S = \{(a_1, \ldots, a_n) \in \mathbb{C}^n | \exists f \in L^2(X^0) \text{ with } A_k f = a_k f\}.$$

Definition 4.1. We say that a finite quotient $C = \Gamma\backslash X$ is **Ramanujan** if the non-trivial simultaneous spectrum of the A_k acting on C is contained in S.

Remark. We remark that the colored adjacency operators do in fact descent to operators on the quotient because our notion of coloured adjacency is preserved by the left action of $\mathrm{GL}_n(\mathbb{Q}_p)$.

We now give a description of the simultaneous spectrum of these operators. Let σ_k be the k^{th} elementary symmetric function. The simultaneous spectrum of the A_k is:

$$S = \{(p^{k(n-k)/2}\sigma_k(z_1, \ldots, z_n))_{k=1\ldots n} | \, |z_i| = 1, z_1 z_2 \cdots z_n = 1\}$$

The projection to the k^{th} component is a simply connected domain with boundary:

$$\{p^{k(n-k)/2}\sigma_k(e^{i\theta}, \ldots, e^{i\theta}, e^{-(n-1)i\theta}) | \theta \in [0, 2\pi]\}$$

Remark. The trivial eigenvectors are the generalizations of taking the constant functions on the quotient. The associated eigenvalues take on the form:

$$\zeta^k p^{k(n-k)/2}\sigma_k(p^{-(n-1)/2}, \ldots, p^{(n-1)/2})$$

for a choice ζ of n^{th} root of unity. These are constructed using 1-dimensional spherical representations of GL_n.

This definition raises the question of how to find interesting subgroups Γ. There are theorems by [LSV] and others which give a large class of examples where these are or are not Ramanujan. The condition which LSV prove is that this is equivalent to certain (ie spherical) subrepresentations of $L^2(\Gamma\backslash X)$ all being tempered.

5. What makes this a "Building"?

We said we were constructing a building, so now the question is, "what makes this a building?". First we should say what a building is. A building is a simplicial complex which is a union of "apartments" satisfying certain combinatorial properties. Rather than list the properties in general I will simply point out what they are in the context of the building we have created.

But first, what are the apartments of X?

Given $e_1, \ldots, e_n$ a basis of $\mathbb{Q}_p^n$ the associated apartment is:

$$\Sigma = \{L = \mathbb{Z}_p a_1 e_1 \oplus \cdots \oplus \mathbb{Z}_p a_n e_n | a_i \in \mathbb{Q}_p^*\}.$$

A lot of the properties of interest concern the automorphism group of the structure, so we should first describe what this is.

$\mathrm{GL}_n(\mathbb{Q}_p)$ acts as automorphisms of X (as a building) this action preserves colored adjacency and the structure of apartments, its kernel is the center $Z(\mathrm{GL}_n(\mathbb{Q}_p)) \simeq \mathbb{Q}_p$. Moreover, this action is transitive.

Remark. We would like for this to be the full automorphism group of X (as a building), it however is not in general. We do however have:

$$Aut(X) = \mathrm{GL}_n(\mathbb{Q}_p)/Z(\mathrm{GL}_n(\mathbb{Q}_p)) + \{\text{Automorphisms of a certain Dynkin Diagram}\}$$

The only automorphism of the dynkin diagram in our case corresponds to $L \mapsto L^\vee$.

(To see why this might be true one should look at the relationship to algebraic structures in the following section, and consider what automorphisms of X could stabilize both an apartment and the vertices of a simplex contained within it).

So what properties do these apartments have that make this a nice structure?

- $\mathrm{GL}_n(\mathbb{Q}_p)$ acts transitively on apartments
  This holds since we can map any basis to any other.
- If $L_1, \ldots, L_r$ are in a single simplex of X there exists a single apartment containing them.
  This holds since we can find a coherent basis for our flag and use that.
- The stabilizer of an apartment acts transitively on its $(n-1)$-simplexes
  To see this involves looking at what is required for a simplex to be in an apartment and what actions stabilize an apartment.
- If a simplex is contained in 2 different apartments there is a map taking one apartment to the other which vertex stabilizes the simplex.
  This involves looking at what is required to be in 2 apartments.

The above properties amount to the statement that what we have constructed is a building.

6. Relationship to Algebraic Structures

We have now constructed a simplicial complex which comes with the action of an algebraic group $GL_n(\mathbb{Q}_p)$. When one has an algebraic object acting on a geometric object one is often interested in what the stabilizers of various structures involved are. In this context they have algebraic significance.

- For $L = \mathbb{Z}_p^n$ we have $Stab(L) = \mathrm{GL}_n(\mathbb{Z}_p) = K$ which is a maximal compact subgroup.
  Consequently we can view $X^0 \simeq \mathrm{GL}_n(\mathbb{Q}_p)/K$. The stabilizers of other latices are just conjugates.

  Remark. The stabilizer of an equivalence class is the center times the stabilizer of a representative.

- The stabilizer of an $(n-1)$-simplex will be a Borel subgroup B. That is for some choice of basis it is the group of upper triangular matrices.
- The stabilizer of an apartment is the normalizer N of a maximal torus T.

  That is to say all we can do is rescale the e_i and permute them which corresponds then to a diagonal matrix (a torus) times S_n (under the permutation matrix realization) which is the torus' normalizer.
- The simultaneous stabilizer of a simplex and an apartment containing it is $B \cap N = T$ a maximal torus.
- The quotient $W = N/T$ thus correspond to automorphisms of the $(n-1)$-simplex which fix an apartment containing it. This group is a coxeter group and acts via reflections on the symplex. It is the "Dynkin Diagram" of this group which essentially determines to what extent $\mathrm{GL}_n(\mathbb{Q}_p)/Z(\mathrm{GL}_n(\mathbb{Q}_p))$ fails to be the automorphism group of the building X.

7. The Spherical Building

We very briefly mention another building which is associated to $\mathrm{GL}_n(\mathbb{Q}_p)$, that is its Spherical Building. Here instead of associating lattices to 0-cells we use for 0-cells the subspaces of $\mathbb{Q}_p^n$ and we use the containment of subspaces to give the structure of 1-cells.

Much of the remainder of the theory is the same. In a certain sense one can describe the Spherical Building as the "Infinite Ends" of the affine building (a compatible sequence of lattices should 'converge' to a subspace). In particular, for $\mathrm{GL}_2(\mathbb{Q}_p)$ the ends of the graph are isomorphic to $P(\mathbb{Q}_p)$ which is the spherical building.