MATH 765 ASSIGNMENT 2

DUE WEDNESDAY OCTOBER 9

1. Let T be an infinite collection of triangles. For any triangle $\tau \in T$, we let $h_\tau = \operatorname{diam}(\tau)$, $|\tau|$ denote the area of τ, and let ρ_τ be the radius of the inscribed circle of τ. Show that the followings are equivalent.
   a) The ratio $h_\tau^2/|\tau|$ is uniformly bounded.
   b) The ratio h_τ/ρ_τ is uniformly bounded.
   c) The minimum angle of τ is uniformly bounded away from 0.

   If any (hence all) of the preceding conditions holds for T, then we say that the collection T is *shape regular* (or *non-degenerate*).
2. Let $\Omega \subset \mathbb{R}^2$ be a polygonal domain and let $\mathscr{P}$ be a family of *conforming* triangulations of Ω. We say that $\mathscr{P}$ is *graded* (or *locally quasi-uniform*, or has the *K-mesh property*), if
$$\sup\left\{\frac{h_\sigma}{h_\tau} : \sigma, \tau \in P,\, \overline{\sigma} \cap \overline{\tau} \neq \varnothing,\, P \in \mathscr{P}\right\} < \infty.$$
   Prove that if $\mathscr{P}$ is shape regular (as the collection $\bigcup_{P\in\mathscr{P}} P$), then it is graded.
3. Let $\Omega \subset \mathbb{R}^n$ be a bounded polyhedral domain. Show that $\mathscr{C}^0$ Lagrange finite element spaces are contained in $W^{k,p}(\Omega)$ for $k \in \{0,1\}$ and any $1 \leq p \leq \infty$, but *not* in $W^{2,p}(\Omega)$ for any $1 \leq p \leq \infty$.
4. Recall that the *Sobolev inequality*
$$\|u\|_{L^q} \leq C|u|_{W^{1,p}}, \qquad u \in \mathscr{D}(\mathbb{R}^n), \tag{$*$}$$
   with some constant $C = C(q)$, is valid for $1 \leq q < \infty$, with $\frac{1}{p} = \frac{1}{q} + \frac{1}{n}$.
   a) By way of a counterexample, show that $(*)$ fails when $\frac{1}{p} \neq \frac{1}{q} + \frac{1}{n}$.
   b) Using the inequality $(*)$, prove
$$\|u\|_{L^q} \leq C\|u\|_{W^{1,p}}, \qquad u \in \mathscr{D}(\mathbb{R}^n), \tag{$**$}$$
   with $C = C(p,q)$, for $1 \leq p \leq q < \infty$, and $\frac{1}{p} \leq \frac{1}{q} + \frac{1}{n}$.
   c) Show that $(**)$ fails when $\frac{1}{p} > \frac{1}{q} + \frac{1}{n}$.
   d) Show that the inequality $(**)$ fails whenever $q < p$.
   e) Show that $(**)$ fails for $q = \infty$ and $p \leq n$ when $n \geq 2$. Is it true in 1d?
5. Let $\Omega \subset \mathbb{R}^n$ $(n \geq 2)$ be a finite union of bounded star-shaped domains. Prove the *Sobolev inequality*
$$\|u\|_{L^q(\Omega)} \leq C\|u\|_{W^{1,p}(\Omega)},$$

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for $1 \le p \le q < \infty$, and $\frac{1}{p} < \frac{1}{q} + \frac{1}{n}$. *Hint*: Use the Young inequality

$$\|f * g\|_{L^q(\mathbb{R}^n)} \le \|f\|_{L^r(\mathbb{R}^n)} \|g\|_{L^p(\mathbb{R}^n)},$$

where $1 + \frac{1}{q} = \frac{1}{p} + \frac{1}{r}$ and $1 \le p, q, r \le \infty$. (Note that the Sobolev inequality is true for the borderline case $\frac{1}{p} = \frac{1}{q} + \frac{1}{n}$ as well, which can be proved for instance by using the Hardy-Littlewood-Sobolev inequality for the Riesz potentials, or by the elementary method due to Gagliardo and Nirenberg.)

6. Let $\tau \subset \mathbb{R}^n$ be a simplex and let $I_\tau : C(\overline{\tau}) \to \mathbb{P}_{d-1}$ be the standard nodal interpolation onto the polynomials of order d. Derive a bound on the interpolation error $\|u - I_\tau u\|_{W^{k,\infty}(\tau)}$ in terms of $h = \operatorname{diam} \tau$ and Sobolev (semi) norms of u. Explicitly state what parameters (k, γ etc.) the constant may depend on.
7. Let $\Omega = (0,1)^2$ be the unit square, and for $j \in \mathbb{N}$, let P_j be the collection of 2^{2j} small squares of side length 2^{-j} tiling up Ω. Denote by $\bar{\mathbb{P}}_{d-1}$ the set of bivariate polynomials of the form $p(x_1)q(x_2)$ with $p, q \in \mathbb{P}_{d-1}$ single variable polynomials. Given $d \in \mathbb{N}$, we define the space S_j of *dyadic splines* as follows:

$$S_j^{d,r} = \{u \in C^r(\Omega) : u|_Q \in \bar{\mathbb{P}}_{d-1} \text{ for each cube } Q \in P_j\}.$$

We also define the *cardinal B-splines* on $\mathbb{R}$ by the recursive formula

$$N^d = N^{d-1} * N^1, \qquad d = 2, 3, \dots,$$

with $N^1 = \chi_{(0,1)}$ the characteristic function of the unit interval.

a) Show that $N^d \in C^{d-2}(\mathbb{R})$, $N^d|_{(k,k+1)} \in \mathbb{P}_{d-1}$ for $k \in \mathbb{Z}$, and $\operatorname{supp} N^d = [0, d]$.

b) We fix d, and define the dyadic cardinal B-splines

$$\phi_{j,k}(x) = N^d(2^j x - k), \qquad j \in \mathbb{N}_0,\ k \in \mathbb{Z},$$

and their tensor product version

$$\phi_{j,\alpha}(x, y) = \phi_{j,\alpha_1}(x)\phi_{j,\alpha_2}(y), \qquad j \in \mathbb{N}_0,\ \alpha \in \mathbb{Z}^2.$$

For $j \in \mathbb{N}_0$, let Φ_j be the collection of those $\phi_{j,\alpha}$ ($\alpha \in \mathbb{Z}^2$) whose supports nontrivially intersect the unit square Ω. Show that Φ_j is a basis of $S_j^{d,d-2}$.

c) From now on we will fix $d = 4$. For each $Q \in P_j$, we define the *Hermite interpolant* $v = H_Q u \in \bar{\mathbb{P}}_3$ for functions $u \in C^1(\overline{\Omega})$ by the following relations

$$\begin{aligned} v(x) &= u(x), \\ \partial_i v(x) &= \partial_i u(x), \quad (i = 1, 2), \\ \partial_1 \partial_2 v(x) &= \partial_1 \partial_2 u(x), \end{aligned}$$

where x runs over the corner points of Q. Since $\dim \bar{\mathbb{P}}_3 = 16$, the polynomial v is well defined. Let us define the global interpolant $H_j u$ by $(H_j u)|_Q = H_Q u$ for each $Q \in P_j$. Show that $H_j u \in S_j^{4,1}$ for $u \in C^1(\overline{\Omega})$.

d) Prove the error estimate

$$\|u - H_j u\|_{W^{k,p}(\Omega)} \le c\, 2^{-j(m-k)} |u|_{W^{m,p}(\Omega)},$$

for $0 \le k \le m \le 4$, $m > \frac{n}{p} + 1$ and $1 \le p \le \infty$. Why are there restrictions on m?