MATH 765 ASSIGNMENT 1

DUE FRIDAY SEPTEMBER 20

1. Let X be a Hilbert space, and let $A : X \to X'$ be a bounded linear operator. Suppose that A is *strictly coercive* in the sense that
$$\langle Au, u\rangle \geq \alpha \|u\|^2 \qquad \text{for} \quad u \in X,$$
with some constant $\alpha > 0$.
   a) Show that for any $f \in X'$, there exists a unique $u \in X$ satisfying $Au = f$. Show also that $\|u\| \leq \alpha^{-1}\|f\|$.
   b) Let $X_h \subset X$ be a a closed linear subspace, and let $f \in X'$. Then the *Galerkin problem* is: Find $u_h \in X_h$ satisfying
$$\langle Au_h, v\rangle = \langle f, v\rangle \qquad \text{for all} \quad v \in X_h.$$
   Show that the Galerkin problem has a unique solution, and that $\|u_h\| \leq \alpha^{-1}\|f\|$. The solution u_h is called the *Galerkin approximation* of u from the subspace X_h.
   c) Show that $u - u_h$ is A-orthogonal to X_h, i.e.,
$$\langle A(u - u_h), v\rangle = 0 \qquad \text{for all} \quad v \in X_h.$$
   This property is called *Galerkin orthogonality*.
   d) Prove that the Galerkin approximation satisfies *Céa's lemma*:
$$\|u - u_h\| \leq \frac{\|A\|}{\alpha} \inf_{v \in X_h} \|u - v\|.$$
2. Show that piecewise quadratics have a nodal basis consisting of values at the nodes x_i together with the midpoints $\frac{1}{2}(x_i + x_{i+1})$. Calculate the stiffness matrix for these elements, corresponding to the problem $-u'' = f$ with $u(0) = u(1) = 0$.
3. For measurable functions u and w defined on the interval $I = (0, 1)$, we say that w is a *weak derivative* of u if
$$\int_I u\varphi' = -\int_I w\varphi, \qquad \text{for all} \quad \varphi \in C^1_c(I).$$
Moreover, for $u, v \in L^p(I)$, we say that v is a *strong L^p derivative* of u if there exists a sequence $\{u_k\} \subset C^1(I)$ such that
$$u_k \to u \quad \text{and} \quad u_k' \to v \quad \text{both in } L^p, \quad \text{as} \quad k \to \infty.$$
Show that each of the two derivatives is unique, if it exists, up to modifications on sets of measure zero. Prove that for $u \in L^p(I)$, both concepts coincide. In other words, show that each of the following statements implies the others.

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a) $v \in L^p(I)$ is a weak derivative of u.
b) $v \in L^p(I)$ is a strong L^p derivative of u.

4. In this exercise we will study Sobolev spaces on the interval $I = (0, 1)$. Let $1 \le p < \infty$, and define the norm
$$\|u\|_{1,p} = \left(\|u\|_{L^p}^p + \|u'\|_{L^p}^p\right)^{1/p},$$
for $u \in C^1(\bar{I})$. Then we define $H^{1,p}(I)$ to be the completion of $C^1(\bar{I})$ with respect to the norm $\|\cdot\|_{1,p}$, and let
$$W^{1,p}(I) = \{u \in L^p(I) : u' \in L^p(I)\},$$
where u' is understood in the weak sense (or in the strong, or in the Beppo Levi sense, since they are all equivalent).
a) Prove the *Meyers-Serrin theorem*: $H^{1,p}(I) = W^{1,p}(I)$.
b) Prove the *Sobolev inequality*
$$\|u\|_{L^\infty} \le 2^{1-1/p}\|u\|_{1,p}, \qquad u \in W^{1,p}(I).$$
c) Recall that an element of $W^{1,p}(I)$ is an equivalence class of functions that differ on sets of measure zero. So one can change the values of a function on a set of measure zero, and it would still correspond to the same element in $W^{1,p}(I)$. Make sense of, and prove the statement that the elements of $W^{1,p}(I)$ are continuous functions.
d) Prove the *Friedrichs inequality*
$$\|u\|_{L^p}^p \le 2^{p-1}\|u'\|_{L^p}^p + 2^{p-1}|u(\xi)|^p, \qquad u \in W^{1,p}(I), \quad \xi \in [0,1].$$
e) Prove the *Poincaré inequality*
$$\|u\|_{L^p}^p \le 2^{p-1}\|u'\|_{L^p}^p + 2^{p-1}\Big|\int_I u\Big|^p, \qquad u \in W^{1,p}(I).$$
f) Let $W_0^{1,p}(I)$ be the closure of $C_c^1(I)$ in $W^{1,p}(I)$. Show that
$$W_0^{1,p}(I) = \{u \in W^{1,p}(I) : u(0) = u(1) = 0\}.$$
g) Show that $\{u \in C^1(\bar{I}) : u'(0) = u'(1) = 0\}$ is dense in $W^{1,p}(I)$.
h) Prove the *Rellich-Kondrashov theorem*: $W^{1,p}(I)$ is compact in $L^p(I)$.
i) Show that $W^{1,1}(I) = AC(\bar{I})$, where $AC(\bar{I})$ is the space of absolutely continuous functions on $[0,1]$.

5. Define the function $f \in L^1(B_1)$ by $f(x) = |x|^r$ where $B_1 = \{x \in \mathbb{R}^n : |x| < 1\}$ and $r > 1 - n$ is a constant. Determine the values of $p \ge 1$ such that $f \in W^{1,p}(B_1)$.
6. Let $u \in C^\infty(\Omega)$ be given in polar coordinates by $u(r,\theta) = r^a \sin(a\theta)$ with
$$\Omega = \{(r,\theta) : r < 1,\, 0 < \theta < \pi/a\},$$
where $a \ge \frac{1}{2}$ is a constant. Determine the values of $p \ge 1$ such that $u \in W^{2,p}(\Omega)$.