Lecture 10
More on real interpolation spaces

§1 Technical facts

Let $f \in X_0$. Suppose that $|\cdot|_{X_1}$ is a semi-norm on X_1, i.e. $\|\cdot\|_{X_1} = \|\cdot\|_{X_0} + |\cdot|_{X_1}$. Recall the K-functional definition (9.1) which we write as K'. We re-define K by

$$K(f,t) = \inf_{g \in X_1} (\|f-g\|_{X_0} + t|g|_{X_1}). \tag{10.1}$$

Clearly, $K(f,t) \leq K'(f,t)$. For $t < a$,

$$t\|g\|_{X_1} = t\|g\|_{X_0} + t|g|_{X_1} \leq a\|f-g\|_{X_0} + a\|f\|_{X_0} + t|g|_{X_1},$$

which makes $K'(f,t) \leq (1+a)K(f,t) + a\|f\|_{X_0}$. It therefore follows that

$$(X_0, \|\cdot\|_{K'}) \hookrightarrow (X_0, \|\cdot\|_X) \hookrightarrow (X_0, \|\cdot\|_{K'}),$$

where the K'-norm is induced by (9.1) (justified in Lemma 9.3) whereas the X-norm is given by $\|\cdot\|_{X_0} + \Phi_{\theta,q}(f)$.

Remark. *It suffices to know $K(f, 2^{-j})$ for nonnegative integers j.*

We have the following technical facts:

- For $q = \infty$,

$$\sup_{0<t<1} t^{-\theta} K(f,t) \leq \sup_{j\in\mathbb{N}} 2^{(j+1)\theta} K(f,2^{-j}) \leq 2^\theta \sup_{j\in\mathbb{N}} 2^{j\theta} K(f,2^{-j}) \tag{10.2}$$

- For $q < \infty$,

$$\int_0^1 [t^{-\theta}K(f,t)]^q \frac{dt}{t} \leq \sum_{j\in\mathbb{N}} 2^{q(j+1)\theta} K(f,2^{-j})^q 2^{-j} 2^j = 2^{q\theta} \sum_{j\in\mathbb{N}} 2^{j\theta q} K(f,2^{-j}). \tag{10.3}$$

  This motivates the following definition. Write $|b|_{\ell_q^\theta} = \|(2^{j\theta}b_j)_j\|_{\ell_q}$,

$$f \in [X_0,X_1]_{\theta,q} \iff \left(K\left(f,2^{-j}\right)\right)_j \in \ell_q^\theta \iff \left(2^{j\theta}K(f,2^{-j})\right)_j \in \ell_q. \tag{10.4}$$

- $[X_0,X_1]_{\theta,q_1} \hookrightarrow [X_0,X_1]_{\theta,q_2}$ if $q_1 < q_2$.
- $[X_0,X_1]_{\theta_1,q_1} \hookrightarrow [X_0,X_1]_{\theta_2,q_2}$ if $\theta_1 > \theta_2$. Notice that there is no specified relation on the q_i's. This is called *lexicographical ordering*. In particular, write $2^{j\theta_2}K(f,2^{-j}) = 2^{-j(\theta_1-\theta_2)}2^{j\theta_1}K(f,2^{-j})$. For $q_1 = \infty$ and $q_2 = 1$,

$$\sum_{j\in\mathbb{N}} 2^{j\theta_2}K(f,2^{-j}) \leq \left[\sup_{j\in\mathbb{N}} 2^{j\theta_1}K(f,2^{-j})\right] \sum_{j\in\mathbb{N}} 2^{-j(\theta_1-\theta_2)},$$

which implies

$$\|\left(2^{j\theta_2}K(f,2^{-j})\right)_j\|_{\ell^1} \le C\|\left(2^{j\theta_1}K(f,2^{-j})\right)_j\|_{\ell^\infty}. \tag{10.5}$$

§2 Examples

Recall the following definition(s):

$$\omega(f,h) = \sup_{|x-y|\le h}|f(x)-f(y)|, \quad \text{(Modulus of continuity).} \tag{10.6}$$

$$|f|_{\mathrm{Lip}} = \sup_{h>0}\omega(f,h) = \sup_{x,y}\frac{|f(x)-f(y)|}{|x-y|}. \tag{10.7}$$

$$|f|_{\mathrm{Lip}\,\alpha} = \sup_{t>0}\frac{\omega(f,t)}{t^\alpha}. \tag{10.8}$$

$$\|f\|_{\mathrm{Lip}\,\alpha} = \|f\|_\infty + |f|_{\mathrm{Lip}\,\alpha}. \tag{10.9}$$

Example 8. Let $X_0 = C(\mathbb{T})$ and $X_1 = C^{0,1}(\mathbb{T}) = \mathrm{Lip}1$. We define the *Steklov average* for a function f by

$$f_h(x) = \frac{1}{h}\int_0^h f(x+s)\,ds. \tag{10.10}$$

We have

$$|f(x)-f_h(x)| \le \frac{1}{h}\int_0^h |f(x+s)-f(x)|\,ds \le \omega(f,h).$$

Clearly, $|f_h(x)| \le \frac{1}{h}\int_0^h |f| \le \|f\|_\infty$. For $0<\delta<h$

$$|f_h(x)-f_h(x+\delta)| \le \frac{1}{h}\int_0^\delta |f(x+s)-f(x+s+h)|\,ds \le \frac{\delta}{h}\omega(f,h).$$

For $\delta > h$,

$$|f_h(x)-f_h(x+\delta)| \le \frac{1}{h}\int_x^{x+h}|f(y)-f(y+\delta)|\,dy \le \frac{1}{h}h\left(\frac{\delta}{h}+1\right)\omega(f,h) \le \frac{2\delta}{h}\omega(f,h).$$

Using those estimates,

$$K(f,t) \le \|f-f_t\|_\infty + t\|f_t\|_{\mathrm{Lip}1} \le \omega(f,t) + t\frac{2}{t}\omega(f,t) = 3\,\omega(f,t).$$

Using the fact that ω satisfies the triangle inequality and that $\omega(f,t) \le 2\|f\|_\infty$, we have for any $g \in \mathrm{Lip}1$,

$$\omega(f,t) \le \omega(f-g,t) + \omega(g,t) \le 2\|f-g\|_\infty + t|g|_{\mathrm{Lip}1},$$

which makes $\omega(f,t) \le 2K(f,t)$. Moreover, if $f \in [C, \mathrm{Lip}1]_{\theta,\infty}$,

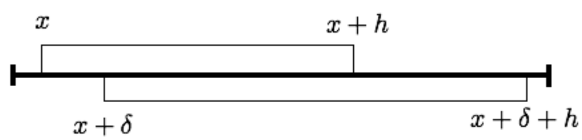
$$\implies K(f,t) \le Ct^\theta \implies \omega(f,t) \le Ct^\theta \implies f \in \mathrm{Lip}\,\theta,$$

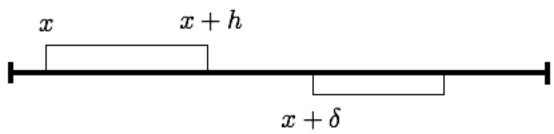
so we may conclude that $[C, \mathrm{Lip}1]_{\theta,\infty} = \mathrm{Lip}\,\theta$. $\diamond$

For the next example, define $B^\alpha_{\infty,q}(\mathbb{T})$ by

$$|f|_{B^\alpha_{\infty,q}} = \left(\int_0^\infty [t^{-\theta}\omega(f,t)]^q \frac{dt}{t} \right)^{1/q}, \tag{10.11}$$

so we see that $[C, \mathrm{Lip}1]_{\theta,q} = B^\theta_{\infty,q}(\mathbb{T})$; set $B^\alpha_{\infty,\infty} = \mathrm{Lip}\,\alpha$.





Example 9. Let $X_0 = C(\mathbb{T})$ and $X_1 = C^1(\mathbb{T})$. Define

$$f_h(x) = \frac{1}{h}\int_x^{x+h} f(y)\,dy = \frac{1}{h}\left[\int_0^{x+h} f(y)\,dy - \int_0^x f(y)\,dy\right],$$

and

$$f_h'(x) = \frac{1}{h}\left[f(x+h) - f(x)\right] \implies |f_h'(x)| \le \frac{\omega(f,h)}{h} \le \omega(f,t) + f\frac{\omega(f,t)}{t} = 2\omega(f,t).$$

We have $K(f,t) \le \|f - f_t\|_\infty + t|f_t|_{C^1}$. Moreover, for all $g \in C^1$, $\left|\frac{g(x)-g(y)}{x-y}\right| = g'(\xi)$ for some ξ which makes $\omega(g,h) \le h\|g'\|_\infty$ and in turn

$$\omega(f,t) \le \omega(f-g,t) + \omega(g,t) \le 2\|f-g\|_\infty + t|g|_{C^1}.$$

It therefore follows $\omega(f,t) \le 2K(f,t)$. Finally,

$$f \in [C, C^1]_{\theta,\infty} \iff K(f,t) \le Ct^\theta \iff \omega(f,t) \le Ct^\theta \iff f \in \mathrm{Lip}\,\theta,$$

The real interpolation space $[C, C^1]_{\theta,\infty} = \mathrm{Lip}\,\theta$ and in general $[C, C^1]_{\theta,q} = B^\theta_{\infty,q}$. $\diamond$