Lecture 9
Peetre's K-method of interpolation

§1 Introduction

Let X_1 and X_0 be Banach spaces with $X_1 \hookrightarrow X_0$. We aim to construct intermediate spaces between X_0 and X_1 with favourable properties. We note that it possible to formulate an interpolation theory only requiring X_0, X_1 be continuously embedded in some bigger space X, where X_0 and X_1 do not satisfy an inclusion.

Definition 9.1. We define the K*-functional* $K(f, t; X_0, X_1)$ for a given function $f \in X_0$ by

$$K(f,t) = \inf_{g \in X_1} \left(\|f - g\|_{X_0} + t\|g\|_{X_1} \right), \quad \big(t > 0\big). \tag{9.1}$$

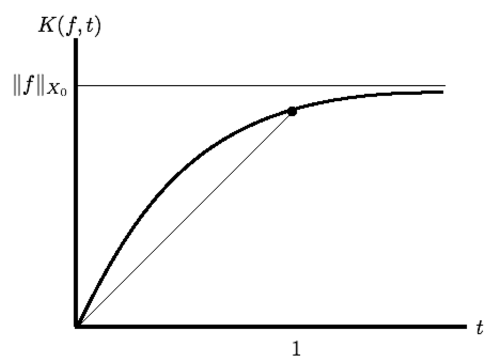
To observe the behaviour of K, take $X_0 = C$ and $X_1 = C^2$ and let $f \in X_0$ have a kink (e.g piecewise linear). See that the *regularization* $\|g\|_{X_1}$ in (9.1) controls the curvature of g. This allows for characterization of an interpolation space satisfying certain restrictions on the curvature, and in turn the smoothness, while remaining sufficiently close to the interpolant f in the X_0 sense. The term $\|f - g\|_{X_0}$ is often called the *fidelity* term.

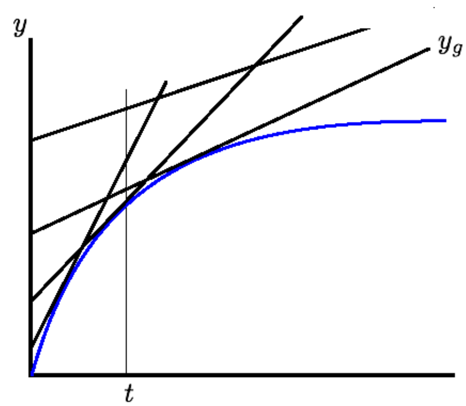
We have the following simple properties:

- If X_1 is dense in X_0, then $K(f, 0) = \inf_{g \in X_1} \|f - g\|_{X_0} = 0$.
- For any $f \in X_0$, $K(f, t) \leq \|f\|_{X_0}$.
- $\min\{1, s\}K(f, 1) \leq K(f, t)$. Indeed,

  $$K(f,t) \leq \|f - g\|_{X_0} - t\|g\|_{X_1} \leq (\|f - g\|_{X_0} + s\|g\|_{X_1}) \max\{1, t/s\},$$

  making $K(f, t) \leq \max\{1, t/s\}K(f, s)$ and result follows when $t = 1$.





Lemma 9.2. *For each fixed* $f_0 \in X_0$*,* $K(f,\cdot)$ *is a nondecreasing, concave and continuous function.*

Proof. Let $t_1 < t_2$. Then $t_1\|g\|_{X_1} \le t_2\|g\|_{X_1}$ implying $K(f,t_1) \le K(f,t_2)$. For concavity, take $y_g(t) = \|f-g\|_{X_0} + t\|g\|_{X_1}$. The envelope generated by all the infimums of y_g is concave. $\square$

§2 Gagliardo diagram

Let $f \in X_0$ be fixed. Consider $\mathbb{R}^2$ as the space of ordered pairs (x_0, x_1). The pair $(x_0, x_1) \in \Gamma \subseteq \mathbb{R}^2$ if and only if there exists a $g \in X_1$ such that $\|f-g\|_{X_0} \le x_0$ and $\|g\|_{X_1} \le x_1$. Let now $\theta \in (0,1)$ and consider the convex combination

$$(\theta x_0 + (1-\theta)y_0, \theta x_1 + (1-\theta)y_0), \quad x_i.y_i \in \Gamma,$$

then by definition there exists a function $g' \in X_1$ satisfying $\|f-g'\|_{X_0} \le y_0$ and $\|g'\|_{X_1} \le y_1$. Therefore if $\tilde{g} = \theta g + (1-\theta)g'$,

$$\|\tilde{g}\|_{X_1} \le \theta x_1 + (1-\theta)y_1,$$

and

$$\|f-\tilde{g}\|_{X_0} \le \theta x_0 + (1-\theta)y_0,$$

which means that Γ is convex. We have:

$$K(f,t) = \inf_{(x_0,x_1)\in\Gamma}(x_0 + tx_1). \tag{9.2}$$

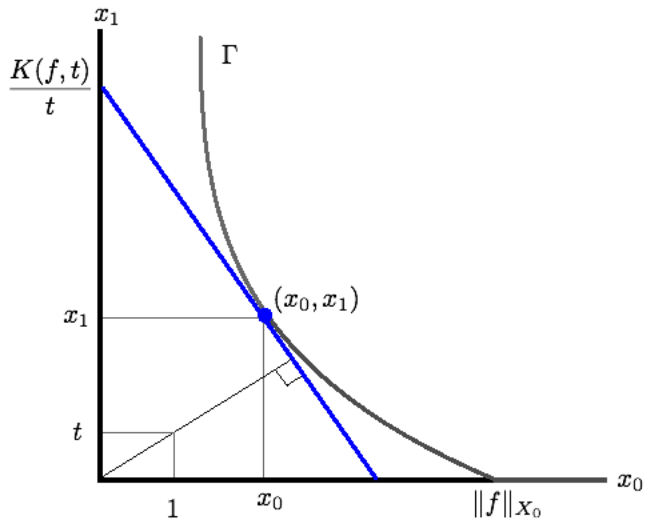


Figure 4: Gagliardo digram

§3 Properties of real interpolation spaces

In this section we will show that for any fixed positive t, $(X_0, \|\cdot\|_{X_0})$ and $(X_0, K(\cdot, t))$ are equivalent. We have the following lemmas:

Lemma 9.3. *For each $t > 0$, $K(\cdot, t)$ is an equivalent norm on X_0.*

Proof. First of all, let $t > 0$ be fixed. $K(\cdot, t)$ satisfies the triangle inequality. Indeed, for any $f_1, f_2 \in X_0$

$$\begin{aligned} K(f_1 + f_2, t) &\leq \|f_1 + f_2 - g_1 - g_2\|_{X_0} + t\|g_1 + g_2\|_{X_1} \\ &\leq \|f_1 - g_1\|_{X_0} + t\|g_1\|_{X_1} + \|f_2 - g_2\|_{X_0} + t\|g_2\|_{X_1}. \end{aligned}$$

Suppose now $K(f, t) = 0$. Then for any $\epsilon > 0$, there exists a $g \in X_1$ such that $\|f - g\|_{X_0} + \|g\|_{X_1} \leq \epsilon$. The nondegeneracy follows from $X_1 \hookrightarrow X_0$. Indeed,

$$\|f\|_{X_0} \leq \|f - g\|_{X_0} + \|g\|_{X_0} \leq \epsilon + C\|g\|_{X_1} \leq \left(1 + \frac{C}{t}\right)\epsilon.$$

To prove the equivalence, recall the second property above. We have

$$\|f\|_{X_0} \leq \|f - g\|_{X_0} + C\|g\|_{X_1} \leq \max\{1, c/t\}(\|f - g\|_{X_0} + t\|g\|_{X_1}),$$

which makes

$$K(f, t) \leq \|f\|_{X_0} \leq \max\{1, c/t\}K(f, t). \tag{$*$}$$

□

Definition 9.4. Define $\Phi_{\theta,q} : X_0 \to \mathbb{R} \cup \{\infty\}$ by

$$\Phi_{\theta,q}(f) = \left(\int_0^\infty [t^{-\theta} K(f,t)]^q \frac{dt}{t} \right)^{1/q}, \quad \big(0 < \theta < 1,\ 1 \leq q \leq \infty\big). \tag{9.3}$$

We define the *real interpolation space*

$$[X_0, X_1]_{\theta,q} = \{f \in X_0 : \Phi_{\theta,q}(f) < \infty\}. \tag{9.4}$$

Observe the following:

- For $q < \infty$,

$$\Phi^q_{\theta,q}(f) \geq \int_0^1 [t^{-\theta} t K(f,1)]^q \frac{dt}{t} + \int_1^\infty [t^{-\theta} K(f,1)]^q \frac{dt}{t}.$$

  On $[0,1]$, $tK(f,1)$ behaves linearly and the integrals are finite if and only if $1 - \theta > 0$ and $\theta > 0$; we require $0 < \theta < 1$.

- For $q = \infty$,

$$\Phi_{\theta,\infty}(f) \geq \max \left\{ \sup_{0 \leq t \leq 1} t^{-\theta} L(f,1), \sup_{t > 1} t^{-\theta} K(f,1) \right\}.$$

  The above is finite if and only if $1 - \theta \geq 1$ and $\theta \geq 0$; we require $0 \leq \theta \leq 1$.

- If $\Phi_{\theta,\infty}(f) < \infty$, then $K(f,t) = \mathcal{O}\left(t^\theta\right)$.
- If $\Phi_{\theta,q}(f) < \infty$ for $q < \infty$, $K(f,t) = o\left(t^\theta\right)$; we gain additional decay.

Lemma 9.5. *The space $[X_0, X_1]_{\theta,q}$ admits a Banach space structure with respect to the norm $\|f\|_{[X_0,X_1]_{\theta,q}} := \Phi_{\theta,q}(f)$. Moreover, $X_1 \hookrightarrow [X_0, X_1]_{\theta,q} \hookrightarrow X_0$.*

Proof. The Banach space structure proof is omitted. For the embedding, $\|f\|_{X_0} \leq CK(f,1) \leq C\Phi_{\theta,q}(f)$ by $(*)$. Conversely, For $f \in X_1$, $K(f,t) \leq \|f\|_{X_0} \leq C\|f\|_{X_1}$ and $K(f,t) \leq t\|f\|_{X_1}$, therefore $K(f,t) \leq \min\{C, t\}\|f\|_{X_1}$. Have

$$K(f,t) \leq \int_0^a [t^{-\theta} t \|f\|_{X_1}]^q \frac{dt}{t} + \int_a^\infty [t^{-\theta} \|f\|_{X_1}]^q \frac{dt}{t} \leq C\|f\|^q_{X_1},$$

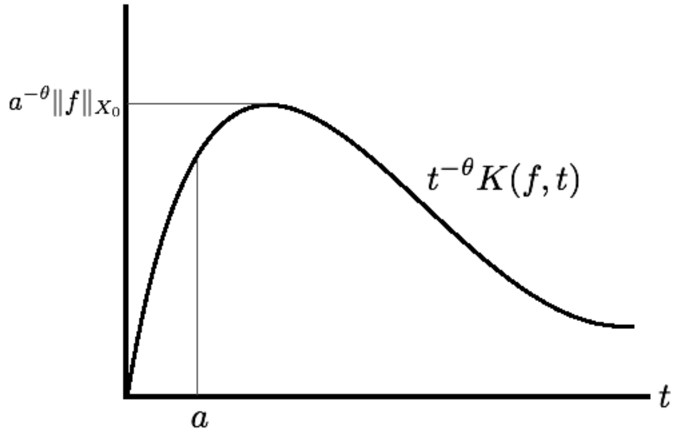
proving $X_1 \hookrightarrow [X_0, X_1]_{\theta,q}$. Now to prove the other embedding,

$$\sup_{t>0} t^{-\theta} K(f,t) \leq \sup_{0<t<a} t^{-\theta} K(f,t) + a^{-\theta} \|f\|_{X_0},$$

which makes

$$\int_0^\infty [t^{-\theta} K(f,t)]^q \frac{dt}{t} \leq \int_0^a [t^{-\theta} K(f,t)]^q \frac{dt}{t} + C\|f\|_{X_0}.$$

$\square$



Remark. Pick $a = 1$ (see behaviour near $t = 1$),

$$\|\xi\|_{\Gamma_q^\theta} = \left(\int_0^1 [t^{-\theta}\xi(t)]^q \frac{dt}{t}\right)^{1/q} \quad \therefore\ f \in X \iff K(f,\cdot) \in \Gamma_q^\theta.$$