Lecture 8
L^p-stability

Let Ω be a bounded polyhedral domain in $\mathbb{R}^n$, let P be a conforming partition of Ω and let $S_P = S^d(P)$ be a Lagrange finite element space. Write

$$h_{min} = \min_{\tau\in P} h_\tau, \quad h_{max} = \max_{\tau\in P} h_\tau, \quad \gamma = \max_{\tau\in P} \gamma_\tau.$$

The direct (Jackson) estimate

$$\inf_{v\in S_P} \|u-v\|_{W^{k,p}(\Omega)} \le C h_{max}^{s-k} |u|_{W^{s,p}(\Omega)}, \quad (0 \le k \le s \le d), \tag{8.1}$$

we have proved

$$\inf_{v\in S_P} \|u-v\|_{W^{k,p}(\Omega)} \le C h_{max}^{d-k} |u|_{W^{m,p}(\Omega)}, \quad (m > n/p), \tag{8.2}$$

where $C = C(\gamma)$. Furthermore, we have proved the inverse (Bernstein) estimates for $v \in S_P$:

$$\|v\|_{L^p(\Omega)} \le C h_{min}^{\frac{n}{p}-\frac{n}{q}} \|v\|_{L^q(\Omega)}, \quad (1 \le q \le p \le \infty), \tag{8.3}$$

and

$$\|v\|_{W^{1,p}(\Omega)} \le C h_{min}^{m-k} \|v\|_{L^p(\Omega)}, \tag{8.4}$$

where $C = C(\gamma)$.

§1 L^p-stability

For $v = \sum_{z\in\mathcal{N}_P} \xi_z\phi_z$, define the quantity

$$[v]_p = \left(\sum_{z\in\mathcal{N}_P} |\xi_z|^p \|\phi_z\|_{L^p}^p\right)^{1/p}. \tag{8.5}$$

Theorem 8.1. *Let $0 < p \le \infty$. Then for any $v \in S_P$*

$$c_1[v]_p \le \|v\|_{L^p} \le c_2[v]_p, \tag{8.6}$$

where $c_1 = c_1(n,d,p,\gamma)$ and $c_2 = c_2(n,d,p)$.

Proof. Let $v = \sum_{z\in\mathcal{N}_P} \xi_z\phi_z$,

$$\|v\|_{L^p} = \sum_{\tau\in P} \|v\|_{L^p}^p = \sum_{\tau\in P} \left\| \sum_{z\in\mathcal{N}_\tau} \xi_z\phi_z \right\|_{L^p(\tau)}^p \le C \sum_{\tau\in P}\sum_{z\in\mathcal{N}_\tau} |\xi_z|^p \|\phi_z\|_{L^p(\tau)}^p, \quad \big(C = C(d)\big),$$

by convexity of $x \mapsto x^p$. Let $\kappa_{z,\tau} = |\xi_z|^p \|\phi_z\|^p_{L^p(\tau)}$, it is clear that $\kappa_{z,\tau} = 0$ if $\tau \cap \operatorname{supp} \phi_z = \emptyset$ and

$$\sum_{\substack{\tau \in P \\ \tau \cap \omega_z \neq \emptyset}} |\xi_z|^p \|\phi_z\|^p_{L^p(\tau)} = |\xi_z| \|\phi_z\|^p_{L^p(\omega_z)}, \quad (\omega_z = \operatorname{supp} \phi_z).$$

We have

$$\|v\|^p_{L^p} \leq C \sum_{\substack{\tau \in P \\ z \in \mathcal{N}_P}} \kappa_{z,\tau} = C \sum_{z \in \mathcal{N}_P} \sum_{\substack{\tau \in P \\ \tau \cap \omega_z \neq \emptyset}} \kappa_{z,\tau} = C \sum_{z \in \mathcal{N}_P} |\xi_z| \|\phi_z\|^p_{L^p(\omega_z)} = C[v]^p_{L^p},$$

where C depends on d. Conversely, we want

$$\sum_{z \in \mathcal{N}_\tau} |\xi_z| \|\phi_z\|^p_{L^p(\tau)} \leq C \left\| \sum_{z \in \mathcal{N}_\tau} \xi_z \phi_z \right\|^p_{L^p(\tau)},$$

but this is true by scaling τ back to σ and C here will depend on γ. □

§2 Conditioning

Consider the following Poisson problem:

$$\begin{cases} -\Delta u = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

Consider the finite element space $S_P = S^d(P) \cap H^1_0(\Omega)$. $\mathcal{N}_P$ only contains the interior nodes. The Galerkin problem (PG) is:

$$a(u_P, v) = \langle f, v \rangle \quad v \in S_P,$$

where $u_P \in S_P$ and $a(\cdot, \cdot)$ is the bilinear form $a(u,v) = \langle \nabla u, \nabla v \rangle$. Solving (PG) is equivalent to solving the linear system $A\xi = \eta$, where

$$A_{yz} = a(\phi_y, \phi_z) = \langle \nabla \phi_y, \nabla \phi_z \rangle, \quad y, z \in \mathcal{N}_P,$$

and

$$\eta_z = \langle f, \phi_z \rangle, \quad u_P = \sum_{z \in \mathcal{N}_P} \xi_z \phi_z, \quad z \in \mathcal{N}_P.$$

We pose the following question: Is the condition number $\kappa = \|A\| \|A^{-1}\|$ reasonable?

Theorem 8.2. *For the matrix A formulated by* (PG)*,*

$$c_1 h_{max}^{-2} \leq \kappa \leq c_2 \left(\frac{h_{max}}{h_{min}} \right)^n h_{min}^{-2}, \quad c_1 > 0. \tag{8.7}$$

Proof. Since A is symmetric positive definite (consequence of coercivity), we have

$$\lambda_{max} = \max_{\xi} \frac{\xi^T A \xi}{\xi^T \xi}, \quad \lambda_{min} = \min_{\xi} \frac{\xi^T A \xi}{\xi^T \xi}.$$

Have $\|\phi_z\|_{L^2}^2 \leq C h_{max}^n$ and $\|\phi_z\|_{L^2}^2 \geq C h_{min}^2$ so by L^2-stability,

$$C h_{min}^n \xi^T \xi \leq \|v\|_{L^2}^2 \leq C \sum_{z \in \mathcal{N}_P} |\xi_z|^2 \|\phi_z\|_{L^2}^2 \leq C h_{max}^n \xi^T \xi,$$

Moreover,

$$\xi^T A \xi = \sum_{y,z \in \mathcal{N}_P} \xi_y a(\phi_y, \phi_z) \xi_z = a(v,v) = \|\nabla v\|_{L^2}^2,$$

so

$$\frac{\|\nabla v\|_{L^2}^2}{\|v\|_{L^2}^2} \cdot C h_{min}^n \leq \frac{\xi^T A \xi}{\xi^T \xi} \leq \frac{\|\nabla v\|_{L^2}^2}{\|v\|_{L^2}^2} \cdot C h_{max}^n.$$

By Bernstein, $\|\nabla v\|_{L^2}^2 \leq h_{min}^{-2} \|v\|_{L^2}^2$ which makes $\lambda_{max} \leq h_{min}^{-2} h_{max}^n$. By Friedrichs, $\|\nabla v\|_{L^2} \geq C \|v\|_{L^2}$ which makes $\lambda_{min} \geq C h_{min}^n$. We arrive at $\kappa \leq C \left(\frac{h_{max}}{h_{min}} \right)^n h_{min}^{-2}$.

For the lower bound, take $v = \phi_z$, $\|\nabla \phi_z\|_{L^2} \geq C h^{-2} h^n$. If $n \geq 2$, choose z belonging to the largest τ so that $\|\nabla \phi_z\|_{L^2}^2 \geq C h_{max}^{n-2}$ so we can conclude

$$\lambda_{max} \geq \|\nabla \phi_z\|_{L^2}^2 \geq C h_{max}^{n-2}.$$

Take $u \in C_c^\infty(\Omega)$ with $u \equiv 1$ on some $\overline{\mathcal{B}} \subset \Omega$ and $v = I_P u$.

$$\|\nabla v\|_{L^2} \leq \|\nabla (v - u)\|_{L^2} + \|\nabla u\|_{L^2} \leq h_{max} |u|_{W^{2,2}} + |u|_{W^{1,2}} \leq C,$$

(observe that $\|\nabla(v-u)\|_{L^2} = |u - I_P u|_{W^{1,2}}$). We see that $\lambda_{min} \leq C h_{max}^n$ which implies

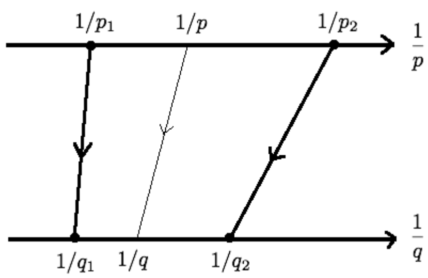
$$\kappa \geq C h_{max}^{-2}.$$

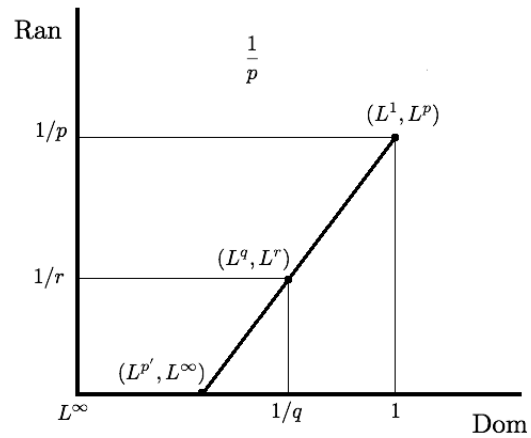
$\square$

§3 Interpolation of function spaces

Recall the Riesz-Thorin convexity theorem:

> Let T be a linear operator such that $T : L^{p_1} \to L^{q_1}$ and $T : L^{p_2} \to L^{q_2}$ are bounded, then $T : L^p \to L^q$ is bounded for
>
> $$\frac{1}{p} = \frac{1-\theta}{p_1} + \frac{\theta}{p_2}, \quad \frac{1}{q} = \frac{1-\theta}{q_1} + \frac{\theta}{q_2}, \quad (0 < \theta < 1).$$





Example 7. Recall $\|u*v\|_p \leq \|u\|_p\|v\|_1$, $\|u*v\|_\infty \leq \|u\|_p\|v\|_q$ where $1/p+1/q = 1$. Define $T_v = u*v$. Then, $T : L^1 \to L^p$ and $T : L^{p'} \to L^\infty$ are bounded, with $1/p + 1/p' = 1$. All pairs in line $\mathcal{L}$ satisfy boundedness. Hence if

$$\frac{1}{p} - \frac{1}{r} = 1 - \frac{1}{q} \implies \frac{1}{r} + 1 = \frac{1}{p} + \frac{1}{q},$$

$T : L^q \to L^r$ is bounded, that is, we have the Young inequality

$$\|u*v\|_r \leq \|u\|_p\|v\|_q.$$

$\diamond$