

MATH 599 PROBLEM SET 4

DUE FRIDAY APRIL 7

- 1) Let M be a Lorentzian manifold of dimension n , and let $\phi : \Sigma \rightarrow M$ be a spacelike submanifold of dimension $n - 2$ embedded into M . We will identify $\phi(\Sigma) \subset M$ with Σ . Let $\gamma : [a, b] \rightarrow M$ be a null geodesic with $\gamma(a) = p \in \Sigma$, $\gamma(b) = q \in M \setminus \Sigma$, and $\gamma'(a) \perp T_p\Sigma$. Such a geodesic will be called a null geodesic *normal to* Σ . Then we define the *null second fundamental form* χ of Σ at p relative to $L = \gamma'(a)$ by

$$\chi(X, Y) = \langle \nabla_X Y, L \rangle,$$

for $X, Y \in \mathfrak{X}(\Sigma)$. Finally, a point $r = \gamma(s)$ is called *conjugate to* (or a *focal point of*) Σ along γ if there exists a nontrivial Jacobi field $X \in \mathfrak{X}^\perp(\gamma)$ along γ such that

- $X(s) = 0$,
- $X(a) \in T_p\Sigma$,
- $\langle \nabla_L X, V \rangle = -\chi(X, V)$ for all $V \in T_p\Sigma$.

Prove that if there exists a point $\gamma(s)$ conjugate to Σ along γ for some $s \in (a, b)$, then there exists a strictly timelike curve connecting Σ and q .

- 2) Consider the Lorentzian manifold $M = I \times \mathbb{R}^3$, equipped with the metric

$$ds^2 = -dt^2 + a(t)^2(dx^2 + dy^2 + dz^2),$$

where $I \subset \mathbb{R}$ is an open interval, and $a(t) > 0$ is the so-called scale factor. This is one of the Robertson-Walker cosmological models. We assume that a perfect fluid with density $\rho = \rho(t) > 0$ and pressure $p = p(t) \geq 0$ is present. That is, we assume that M satisfies the Einstein field equations with the stress-energy tensor given in its *contravariant form* with respect to the coordinate frame $(\partial_t, \partial_x, \partial_y, \partial_z)$ by

$$S = \begin{pmatrix} \rho + p & 0 & 0 & 0 \\ 0 & p & 0 & 0 \\ 0 & 0 & p & 0 \\ 0 & 0 & 0 & p \end{pmatrix}.$$

- (a) Show that the vector field $T = \partial_t \equiv -(dt)^\#$ is a timelike geodesic congruence, and compute its vorticity ω , shear σ , and expansion scalar θ . We take T as future directed.
- (b) Write down the Raychaudhuri equation, and deduce that the scale factor $a(t)$ cannot be constant. Assuming that $\theta(c) > 0$ for some $c \in I$, and assuming that I is large enough, conclude that there is a point $t < c$ at which $\theta(t)$ becomes $-\infty$.
- (c) From the t -component of the contracted Bianchi identity, derive an equation for $\dot{\rho}$. Assuming the model $p = \kappa\rho$ with $\kappa > -1$ a constant, solve this equation for ρ in terms of a . Then substitute ρ into the Raychaudhuri equation, and obtain a solution $a(t)$.
- (d) Choose I as large as possible, so that $a(t)$ remains positive. Show that the resulting Lorentzian manifold M is inextendible, and that all causal geodesics are incomplete.

- 3) Let S be a subset of a time-oriented Lorentzian manifold M . Prove the following.
- (a) $I^+(I^+(S)) = I^+(S)$.
 - (b) $I^+(\overline{S}) = \overline{I^+(S)}$.
 - (c) $J^+(S) \subset \overline{I^+(S)}$.
 - (d) $I^+(S) = \text{int}(J^+(S))$.