MATH 599 PROBLEM SET 3

DUE WEDNESDAY MARCH 15

In this problem set, we will study when geodesics maximize the proper time among timelike curves connecting a hypersurface and a point. Let M be a Lorentzian manifold (of dimension n), and let $\phi : H \to M$ be a spacelike hypersurface (of dimension $n-1$) embedded into M. We will identify $\phi(H) \subset M$ with H. Then $\mathfrak{X}(\phi)^\top$ can be identified with $\mathfrak{X}(H)$, and we have the (pointwise) orthogonal decomposition

$$\mathfrak{X}(\phi) = \mathfrak{X}(\phi)^\top \oplus \mathfrak{X}(\phi)^\perp = \mathfrak{X}(H) \oplus \mathfrak{X}(\phi)^\perp,$$

where $\mathfrak{X}(\phi)^\perp$ is the space of vector fields along ϕ that are pointwise orthogonal to H. We denote by $P^\perp : \mathfrak{X}(\phi) \to \mathfrak{X}(\phi)^\perp$ the orthogonal projection onto $\mathfrak{X}(\phi)^\perp$. Thus, if $N \in T_pM$ is a nonzero vector at $p \in H$ satisfying $N \perp H$, then

$$(P^\perp X)_p = \frac{\langle X, N\rangle_p}{\langle N, N\rangle_p} N \qquad \text{for} \quad X \in \mathfrak{X}(\phi).$$

Suppose that $q \in M \setminus H$, and consider the set $C(H,q)$ of all timelike curves joining H and q. Given a curve $\gamma \in C(H,q)$, a smooth map $\omega : (-\varepsilon, \varepsilon) \times [a,b] \to M$ is called a *deformation of* γ, if $\omega(0,\cdot) = \gamma$, $\omega(\cdot, a) \in H$, $\omega(\cdot, b) = q$, and each "longitudinal" curve $\gamma_s = \omega(s,\cdot)$ is timelike. We set $X = \omega_* \partial_s \in \mathfrak{X}(\omega)$ and $T = \omega_* \partial_t \in \mathfrak{X}(\omega)$, where s and t are the Cartesian coordinates in the rectangle $(-\varepsilon,\varepsilon) \times [a,b]$. Note that $X(s,b) = 0$ and $X(s,a) \in T_{\omega(s,a)}H$ for all s. We will also use X and T to denote the *variation field* $X|_{s=0} \in \mathfrak{X}(\gamma)$ and the *velocity field* $T|_{s=0} \in \mathfrak{X}(\gamma)$, respectively. Hopefully it will not lead to confusion. Recall that the proper time of γ_s is given by

$$\tau(s) = \tau(\gamma_s) = \int_a^b |T| dt = \int_a^b \sqrt{-\langle T, T\rangle} dt.$$

1) Compute the first variation $\tau'(s)$, and show that it depends only on the variation field X and the velocity field T along γ. Assuming that γ is parameterized by proper time as $|T| = 1$, show that $\tau'(0) = 0$ for all variation fields X with $X(b) = 0$ if and only if γ is a geodesic and $T(a) \perp H$.
2) We define the *second fundamental form* $\mathrm{I\!I} : \mathfrak{X}(H) \times \mathfrak{X}(H) \to \mathfrak{X}(\phi)^\perp$ of H by

$$\mathrm{I\!I}(X,Y) = P^\perp \nabla_X Y.$$

   It is clear that $\mathrm{I\!I}$ is bilinear. Prove the following, and conclude in particular that the second fundamental form is tensorial in each of its arguments.
   (a) $\langle \mathrm{I\!I}(X,Y), N\rangle = -\langle \nabla_X N, Y\rangle$ for $N \in \mathfrak{X}(\phi)^\perp$.
   (b) $\mathrm{I\!I}(X,Y) = \mathrm{I\!I}(Y,X)$.
3) Let $\gamma : [a,b] \to M$ be a timelike geodesic with $\gamma(a) = p \in H$, $\gamma(b) = q \in M \setminus H$, and $T(a) \perp H$. Such a geodesic will be called a timelike geodesic *normal to* H. For any deformation of γ, show that

$$\tau''(0) = I(X,X) - \langle \nabla_T X, X\rangle_p,$$

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where I is the index form as defined in class. Show also that the following hold.
(a) $X(b) = 0$.
(b) $X(a) \perp T(a)$, that is, $X(a) \in T_pH$.
(c) $\langle \nabla_T X, V\rangle_p = -\langle \mathrm{II}(X, V), T\rangle_p$ for all $V \in T_pH$.
We call any $X \in \mathfrak{X}(\gamma)$ satisfying these 3 conditions an *H-proper variation*. Moreover, we say that q is *conjugate to H along* γ if there exists a nontrivial Jacobi field along γ that is also an H-proper variation.

4) Consider a timelike geodesic γ normal to H. Let

$$I_H(X, Y) = I(X, Y) - \langle \nabla_T X, Y\rangle_p \qquad \text{for} \quad X, Y \in \mathfrak{X}(\gamma),$$

and prove the following.
(a) For H-proper variations X and Y, we have

$$I_H(X^\perp, Y^\perp) = I_H(X, Y) = I(X, Y) + \langle \mathrm{II}(X, Y), T\rangle_p,$$

where $X \mapsto X^\perp$ is the orthogonal projection onto $\mathfrak{X}(\gamma)^\perp$.
(b) If H has no conjugate point along γ on $(a, b]$, then $I_H(X, X) < 0$ for any nonzero H-proper variation X in $\mathfrak{X}(\gamma)^\perp$.

5) In the same setting, prove that if H has a conjugate point along γ at the parameter value $c \in (a, b)$, there exists an H-proper variation X in $\mathfrak{X}(\gamma)^\perp$ such that $I_H(X, X) > 0$.

6) Show that if γ is a timelike geodesic normal to H, and if X is an H-proper variation, then there is a deformation of γ with its variation field equal to X.