MATH 599 PROBLEM SET 2

DUE FRIDAY FEBRUARY 24

1. Let M be a manifold equipped with an affine connection. Let $p \in M$, and $V, W \in T_pM$. Compute the Jacobi fields (at $s = 0$) for the following geodesic variations.
   (a) $\gamma_s(t) = \exp_p(t(V + sW))$.
   (b) $\gamma_s(t) = \exp_{\eta(s)}(tV_{\eta(s)})$, where $\eta(s) = \exp_p(sW)$ and $V_{\eta(s)} \in T_{\eta(s)}M$ is the parallel transported version of V along η.
2. Let M be equipped with a torsion-free connection ∇, and let $X, Y, Z \in \mathfrak{X}(M)$. Prove the following identities.
   (a) $R(X, Y)Z + R(Y, Z)X + R(Z, X)Y = 0$.
   (b) $\nabla_X R(Y, Z) + \nabla_Y R(Z, X) + \nabla_Z R(X, Y) = 0$.
3. Let M be a pseudo-Riemannian manifold, and let $X, Y, Z, W \in \mathfrak{X}(M)$. Prove the following identities.
   (a) $\langle R(X, Y)Z, W\rangle = -\langle R(X, Y)W, Z\rangle$.
   (b) $\langle R(X, Y)Z, W\rangle = \langle R(Z, W)X, Y\rangle$.
   (c) $\mathrm{div}\mathrm{Ric} \equiv \mathrm{tr}_g \nabla \mathrm{Ric} = \frac{1}{2}\mathrm{d}R$, where $R = \mathrm{tr}_g\mathrm{Ric}$ is the Ricci scalar, and tr_g denotes the trace with respect to the metric.
4. Let $\rho : \mathbb{R} \to \mathbb{R}$ be a smooth function satisfying $\rho(0) = 0$ and $\rho(x) > 0$ for all $x > 0$. Consider the metric
$$ds^2 = \frac{dx^2 + dy^2}{\rho(y)},$$
   in the upper half plane $\mathbb{R}^2_+ = \{(x, y) \in \mathbb{R}^2 : y > 0\}$.
   (a) Compute the Levi-Civita connection, Riemann curvature, Ricci tensor, and the scalar curvature of this metric.
   (b) Identify the geodesics in the cases $\rho(y) = y$ and $\rho(y) = y^2$.
5. Let local coordinates (t, r, θ, ϕ) be given on a 4-manifold M, and consider the form
$$ds^2 = -e^{2a}dt^2 + e^{2b}dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2),$$
   for the metric, where $a = a(r)$ and $b = b(r)$ are functions of r only. To work through this exercise, you might need to use additional resources (books, videos etc.) on geodesics in an exterior Schwarzschild spacetime.
   (a) Compute the Levi-Civita connection, curvature, Ricci tensor, scalar curvature, and the Einstein tensor for this metric. You can compute directly or use a moving frame, cf. [MTW, page 360].
   (b) Find all functions a and b such that the metric satisfies the Einstein field equations in vacuum. In the following we fix these forms for a and b.
   (c) Derive the geodesic equation, and show that all geodesics tangent to the plane $\theta = \frac{\pi}{2}$ stay in the same plane.
   (d) Compute the bending of light rays near the Sun.
   (e) Derive a formula for the perihelion advance of Mercury.

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