SPACETIME SINGULARITIES AND GRAVITATIONAL COLLAPSE

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Abstract. In this course project, we study the gravitational collapse and spacetime singularities.

1. Introduction

Our main goal in this course project is to study the gravitational collapse and naked singularities. If a pressure gradient is not sufficiently strong, a body can continue collapsing due to its self-gravity. This phenomenon is called *gravitational collapse*. The astrophysical relevance of gravitational collapse is now robust. And now we have observational/experimental evidence for massive/supermassive black holes. Note that it is well-understood by physicists that there exists an upper limit to the maximum possible mass of a spherical body of cold nuclear matter. We also learn from class that black holes may have formed from cosmological perturbations in the early stage of the universe.

The singularity theorems state that there exists space-time singularities in generic gravitational collapse. There are actually two kinds of singularities: (i): *curvature singularities* (ii): *coordinate singularities* and we will discuss in details in the case of *Schwarzschild solution* (spherically symmetry solution in vacuum). As we will see, at the curvature singularities, the smoothness of the spacetime metric is lost. Or we can interpret in a more precise way, the curvature singularity is not regraded as a *point* on the spacetime manifold, but the *boundary* of the manifold. Namely, it is *not* on the manifold. In classical physics, we have *no* problem with that since we implicitly assume the smoothness of the metric *at the beginning*. But in the theory of *general relativity*, we need to deal with this kind of problem and seek a reasonable physical/mathematical explanation for that.

When studying spacetime singularities, our main philosophy is the following:
(i) : Solving for the general form the metric.
(ii) : Analyzing the type of singularities.
(iii) : Using coordinate transformation to remove the coordinate singularities.
(iv) : Drawing *suitable* diagram to visualize the *event horizon, black hole, singularities, gravitational collapse* etc.

Contents

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2. Schwarzschild metric

Let us recall our setting, consider a 4-dimensional spacetime manifold M with local coordinates (t, r, θ, ϕ) and the metric

$$ds^2 = -e^{2a(r)}dt^2 + e^{2b(r)}dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)$$

where $a(r)$ and $b(r)$ are the functions of r only.We calculate the Levi-Civita connection, curvature, Ricci tensor, scalar curvature and plug in the Einstein field equations in vacuum, which is

$$R_{\mu\nu} = 8\pi G\big(T_{\mu\nu} - \frac{1}{2}Tg_{\mu\nu}\big) = 0$$

Then we get the Schwarzschild solution

$$ds^2 = -\left(1 + \frac{C}{r}\right)dt^2 + \left(1 + \frac{C}{r}\right)^{-1}dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)$$

Remark 2.1. (1): The Schwarzschild solution is asymptotically flat. Take $r \to \infty$, the Schwarzschild metric reduces to Minkoswi metric.

(2): We want to determine the value of C in the above formula. By the first remark, since the curvature vanishes as $r \to \infty$, the behaviour of test body in the Schwarzschild solution with the parameter C agrees with the behaviour of a test body in the Newtonian gravitational field. After the simple transformation, we can write Newtonian gauge in the form of

$$ds^2 = -(1 + 2\Psi)dt^2 + (1 - 2\Psi)\delta_{ab}dx^a dx^b$$

Thus, we have

$$\begin{cases} -1 - \frac{C}{r} = -1 - 2\Psi \\ 1 + \frac{C}{r} \approx 1 \approx 1 - \frac{C}{r} = 1 - 2\Psi \end{cases}$$

where Ψ is the potential

$$\Psi = -\frac{GM}{r}$$

So if we set

$$C = -2GM$$

These two will agree, hence we get the final form of the Schwarzschild metric

$$ds^2 = -\left(1 - \frac{2GM}{r}\right)dt^2 + \left(1 - \frac{2GM}{r}\right)^{-1}dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)$$

where M is the Newtonian mass measured at large distances $r \gg 2GM$ and G is the Gravitational constant.

(3) The Birkhoff's Theorem says that the Schwarzschild solution is the unique spherical symmetry solution of vacuum Einstein Equation.

(4) As we can see, there are two singularities. $r = 0$ and $r = r_s = 2GM$.

Question 2.2. *(1) Where does these singularities come from?*

(2) Can we remove these singularities to make our theory perfect?

To answer the first question, we want to introduce two kinds of singularities. The first one is called the *Curvature Singularities*.

Definition 2.3. The curvature singularities is a point when the *scalar curvature* blows up/diverges.

Note that here the scalar curvature is not necessarily the Ricci scalar $R = g^{ab}R_{ab}$, we can construct other higher-order scalar curvature such as $R^{ab}R_{ab}$, $R^{abcd}R_{abcd}$ or even $R_{abcd}R^{cdef}R_{ef}{}^{ab}$ and so on. If any of these scalars goes to infinity as we approach this point, we regard this as a singularities of the curvature. We also need to check that this kind of points can be reached by travelling a finite distance along the curve.

Example 2.4. Going back to Schwarzschild spacetime, we compute the quantity

$$R^{abcd}R_{abcd} = \frac{48G^2M^2}{r^6}$$

In fact, this quantity is called *Kretschmann scalar*.

Note that this Kretschmann invariant is crucial in detecting the singularities.

Example 2.5. In Kerr Spacetime, the stationary and axially symmetric Kerr metric is interpreted as describing a spinning black hole. The Kerr line element is given as

$$ds^2 = -\left(1 - \frac{2mr}{\rho^2}\right)dt^2 - \frac{4mar\sin^2\theta}{\rho^2}dtd\varphi + \frac{\Sigma}{\rho^2}\sin^2\theta d\varphi^2 + \frac{\rho^2}{\Delta}dr^2 + \rho^2 d\theta^2$$

or equivalently,

$$ds^2 = -\frac{\rho^2\Delta}{\Sigma}dt^2 + \frac{\Sigma}{\rho^2}\sin^2\theta(d\varphi - \omega dt)^2 + \frac{\rho^2}{\Delta}dr^2 + \rho^2 d\theta^2$$

where

$$\begin{cases} \rho^2 = r^2 + a^2\cos^2\theta \\ \Delta = r^2 - 2mr + a^2 \\ \Sigma = (r^2 + a^2)^2 - a^2\Delta\sin^2\theta \\ \omega = \frac{2mar}{\Sigma} \end{cases}$$

Here m is the mass and a is the angular momentum per unit mass. In this case, the Kretschmann scalar is

$$R_{abcd}R^{abcd} = \frac{48m^2(r^2 - a^2\cos^2\theta)(\rho^4 - 16a^2r^2\cos^2\theta)}{\rho^{12}}$$

As from this, we can see that there is a real singularities at $\rho = 0$.

The other kind of singularities is so called *coordinate singularities*. By coordinate singularities, we mean that this kind of singularities come from our *specific coordinate system*, not from the *manifold*. A typical example is as follows:

Example 2.6. In $\mathbb{R}^2$, the metric under the polar coordinates is presented by

$$ds^2 = dr^2 + r^2 d\theta^2$$

The inverse component $g^{\theta\theta}$ of the metric is $\frac{1}{r^2}$, which is not at the origin, hence we say this metric is *singular*. But this point is definitely on $\mathbb{R}^2$, hence this is a coordinate singularity.

Remark 2.7. In fact, it is indeed possible to make a suitable coordinate transformation so that our system is better behaved at $r = 2GM$. The idea is roughly the following, suppose we fix θ and ϕ constant and let $ds^2 = 0$, then

$$ds^2 = 0 = -\left(1 - \frac{2GM}{r}\right)dt^2 + \left(1 - \frac{2GM}{r}\right)^{-1} dr^2$$

This implies

$$\frac{dt}{dr} = \pm\left(1 - \frac{2GM}{r}\right)^{-1} \tag{2.1}$$

So we see the problem with current coordinates is that $\frac{dt}{dr} \to \infty$ along null geodesics when $r \to 2GM$. Consequently, $\frac{dr}{dt} \to 0$ so the progress in the r direction becomes slower and slower with respect to the coordinate time t. We want to replace t with a coordinate that moves more slowly along the null geodesics. We can explicitly solve the condition (2.1) characterizing radial null curves to obtain

$$t = \pm r^* + C'$$

for some C' constant. And the *tortoise coordinate* r^* is defined by

$$r^* = r + 2GM \ln\left(\frac{r}{2GM} - 1\right) \qquad \text{where} \qquad r \geq 2GM$$

So in terms of the tortoise coordinate, the Schwarzschild metric becomes

$$ds^2 = \left(1 - \frac{2GM}{r}\right)\big(-dt^2 + dr^{*2}\big) + r^2(d\theta^2 + \sin^2\theta d\phi^2) \tag{2.2}$$

where you can think of r as a function of r^*. In this coordinate transformation, the price we pay is to push $r = 2GM$ to infinity.

Next, if we let

$$\begin{cases} v = t + r^* \\ u = t - r^* \end{cases}$$

Then, by coordinate change, we are going to have v and u are constants. Thus,

$$dv = dt + dr^* \qquad \text{and} \qquad dr^* = \left(1 + 2GM\frac{1}{\frac{r}{2GM} - 1} \cdot \frac{1}{2GM}\right)dr \tag{2.3}$$

Plug (2.3) into (2.2) and we get the *Eddington-Finkelstein coordinates*

$$ds^2 = -\left(1 - \frac{2GM}{r}\right)dv^2 + (dvdr + drdv) + r^2(d\theta^2 + \sin^2\theta d\phi^2)$$

Again, we solve the radial null curve, which is

$$\left(1 - \frac{2GM}{r}\right)\left(\frac{dv}{dr}\right)^2 = 2\left(\frac{dv}{dr}\right) \implies \frac{dv}{dr} = \begin{cases} 0 & \text{(ingoing)} \\ 2\left(1 - \frac{2GM}{r}\right)^{-1} & \text{(outgoing)} \end{cases}$$

Remark 2.8. (1): Then we can indeed plot the graph on t-r, t-r^*, v-r and t^*-r, where $t^* = v - r$.
(2): We can see that initially $r \geq 2GM$, but by our diagram, the light cone behaves well at $r = 2GM$. Also the surface is at a finite coordinate value. Hence there is no problem in tracing the paths of null or timelike particles past the surface. On the other hand, as shown in the graph, something interesting is certainly going on.

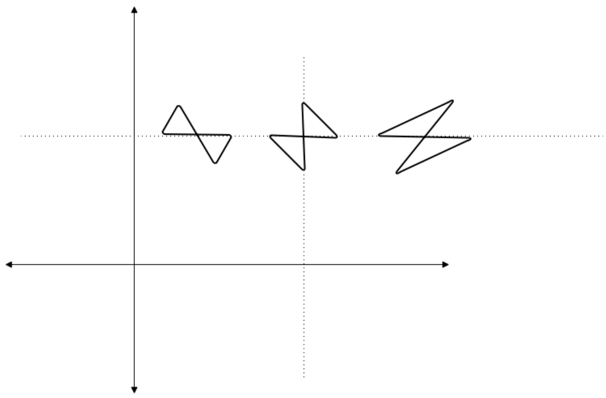

FIGURE 1. $v - r$ graph

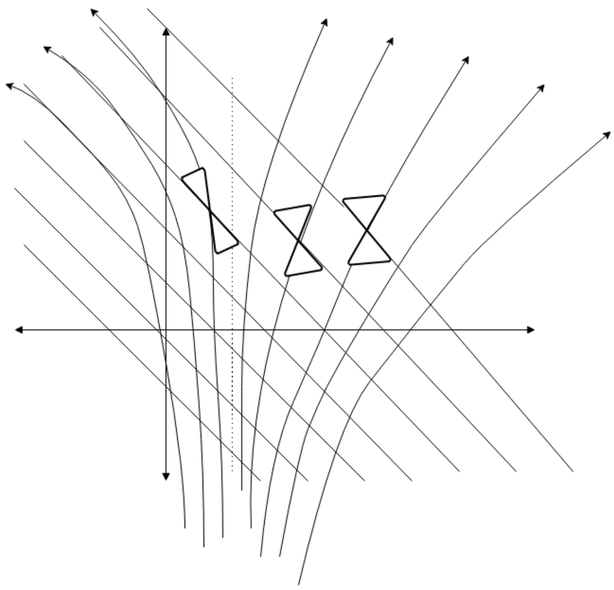

FIGURE 2. $t^* - r$ graph

As we can see from the formula or the graph, once a particle pass through the surface $r = 2GM$, it can never come back and escape to infinity. By this fact, we define the *event horizon* to be a surface past which can never escape to infinity. Of course, in Schwarzschild spacetime, the event horizon is defined at $r = 2GM$.

Since nothing can escape the event horizon, it is impossible for us to see from the inside, we thus call this region *black hole*. A black hole is simply a region of spacetime separated from infinity by an event horizon.

Remark 2.9. The concept about the event horizon is somehow a global one. The location of the horizon is a statement about the spacetime as a whole, not something you could determine just by knowing the geometry at that location. We will see it in some more general spacetime.

3. Extended Schwarzschild Solution

So far we have done a good job in analyzing the coordinates and getting some useful information. But by our coordinate transformation, mathematically, we still

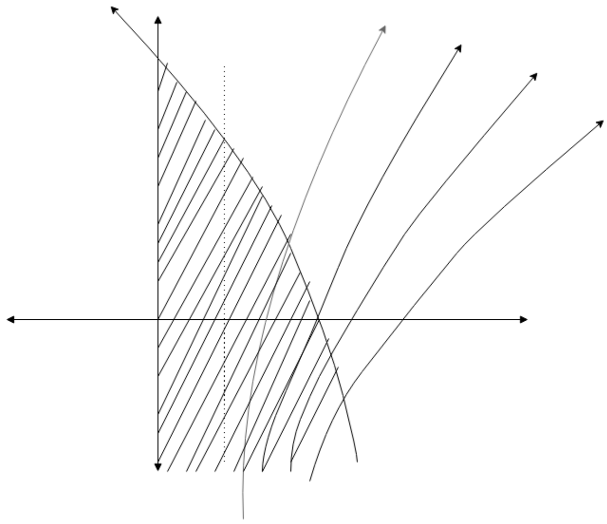

FIGURE 3. Gravitational collapse

have the constraints $r \geq 2GM$. Hence our theory is not yet perfect since our chosen coordinates didn't cover the entire manifold.($r \leq 2GM$)

Our idea of seeking a better coordinate is as follows: we use both u and v at once(in place of t and r). Do a little bit of calculation, we get

$$ds^2 = -\frac{1}{2}\left(1 - \frac{2GM}{r}\right)(dvdu + dudv) + r^2(d\theta^2 + \sin^2\theta d\phi^2)$$

with r is defined implicitly in terms of v and u by

$$\frac{1}{2}(v - u) = r + 2GM \ln\left(\frac{r}{2GM} - 1\right)$$

Remark 3.1. In our new coordinates, our event horizon $r = 2GM$ is infinitely far away.(at either $v = -\infty$ or $u = +\infty$).

So we want to pull these points into the finite values: a good option is to take

$$\begin{cases} v' = e^{\frac{v}{4GM}} \\ u' = -e^{-\frac{u}{4GM}} \end{cases}$$

which in terms of the (t, r) system is

$$\begin{cases} v' = \left(\frac{r}{2GM} - 1\right)^{\frac{1}{2}} e^{\frac{r+t}{4GM}} \\ u' = -\left(\frac{r}{2GM} - 1\right)^{\frac{1}{2}} e^{\frac{r-t}{4GM}} \end{cases}$$

Now, in (v', u', θ, ϕ) system, our Schwarzschild metric is

$$ds^2 = -\frac{16G^3M^3}{r} e^{-\frac{r}{2GM}}(dv'du' + du'dv') + r^2(d\theta^2 + \sin^2\theta d\phi^2)$$

In this metric every metric coefficients behaves very well at the event horizon. Also, both of the partial derivatives $\frac{\partial}{\partial v'}$ and $\frac{\partial}{\partial u'}$ are null vectors. There is nothing wrong with this, but we are somehow more comfortable in working in a system where coordinate is timelike and the others are spacelike. Thus we define

$$\begin{cases} T = \frac{1}{2}(v' + u') = \left(\frac{r}{2GM} - 1\right)^{\frac{1}{2}} e^{\frac{r}{4GM}} \sinh\left(\frac{t}{4GM}\right) \\ R = \frac{1}{2}(v' - u') = \left(\frac{r}{2GM} - 1\right)^{\frac{1}{2}} e^{\frac{r}{4GM}} \cosh\left(\frac{t}{4GM}\right) \end{cases}$$

In term of this substitution, our final metric becomes

$$ds^2 = \frac{32G^3M^3}{r} e^{-\frac{r}{2GM}} (-dT^2 + dR^2) + r^2(d\theta^2 + \sin^2\theta d\phi^2) \tag{3.1}$$

where r is defined implicitly from

$$T^2 - R^2 = \left(1 - \frac{r}{2GM}\right) e^{\frac{r}{2GM}} \tag{3.2}$$

The coordinates (T, R, θ, ϕ) are known as *Kruskal coordinates.*

Remark 3.2. (1): Note that Kruskal coordinates have a lot of nice properties, the redial null curves look like in the flat space:

$$T = \pm R + \text{constant}$$

(2): Another nice property is that the event horizon $r = 2GM$ is not infinitely far away. It is defined by

$$T = \pm R \tag{3.3}$$

consistent with it being a null surface. More generally, we can consider the surfaces where $r =$ constant. From (3.2) these satisfy

$$T^2 - R^2 = \text{constant}$$

Thus they appear as hyperbola in the R-T plane. Furthermore, the surfaces of constant t are given by

$$\frac{T}{R} = \tanh\left(\frac{t}{4GM}\right) \tag{3.4}$$

which defines straight lines through the origin with slope $\tanh(\frac{t}{4GM})$. Note that as $t \to \pm\infty$, (3.4) becomes (3.3). Therefore, $t = \pm\infty$ represents the same surface as $r = 2GM$.

(3): Our coordinates pair (T, R) are allowed to range over all the values without hitting the real singularity at $r = 0$; Hence the allowed region is therefore

$$\begin{cases} -\infty \leq R \leq \infty \\ T^2 < R^2 + 1 \end{cases} \tag{3.5}$$

(4): By the previous remark, we can draw a spacetime diagram in the T-R plane(with θ and ϕ suppressed). This well-known diagram is called *Kruskal diagram.*

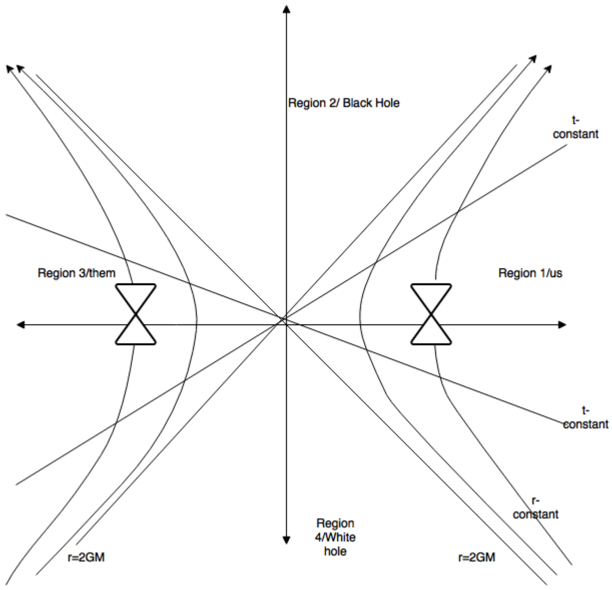

FIGURE 4. Kruskal diagram

4. Conformal Transformation

The ideal of *conformal transformation* is to make a local change of scale. In spacetime, the distance are measured by a metric, so such transformation are done by multiplying a a spacetime-dependent(non-vanishing) function:

$$\tilde{g}_{\mu\nu} = \omega^2(x) g_{\mu\nu}$$

or equivalently,

$$\tilde{ds}^2 = \omega^2(x) ds^2$$

for some non-vanishing function $\omega(x)$. (Here x denote the collection of spacetime coordinates x^μ). Consequently, we have the inverse conformal transformation $g_{\mu\nu} = \omega^{-2}(x)\tilde{g}_{\mu\nu}$.

Notation 4.1. On each tangent space, we set up a basis of four vectors $\{\hat{e}_{(\mu)}\}$, with $\mu \in \{0, 1, 2, 3\}$ as usual. Then any abstract vector A can be written as a linear combination of basis vectors:

$$A = A^\mu \hat{e}_{(\mu)}$$

where A^μ are *components* of the vector A. More often, we just write the vector A^μ.

Lemma 4.2. *Null curves are left invariant by conformal transformations.*

Proof. Suppose we have a null curve $x^\mu(\lambda)$. Then $x^\mu(\lambda)$ is null if and only if its tangent vector $\frac{dx^\mu}{d\lambda}$ is null. Namely,

$$g_{\mu\nu} \frac{dx^\mu}{d\lambda} \frac{dx^\nu}{d\lambda} = 0$$

Then is conformally-related metric we have is

$$\tilde{g}_{\mu\nu} \frac{dx^\mu}{d\lambda} \frac{dx^\nu}{d\lambda} = \omega^2(x) g_{\mu\nu} \frac{dx^\mu}{d\lambda} \frac{dx^\nu}{d\lambda} = 0$$

$\square$

5. Causality

The goal of this section is to recall the definition and notation of the *Causality.*

Definition 5.1. A *causal curve* is the curve which is timelike or null everywhere.

Consequently, given any subset S of a manifold M, we define the *causal future* of S by the set of points that can be reached from S by following a future-directed causal curve. It is denoted by $J^+(S)$.

Similarly, we define the *chronological future* by the set of points that can be reached by following a future-directed timelike curve. It is denoted by $I^+(S)$.

Remark 5.2. (1): A curve of zero is causal but not chronological. Therefore, we always have that $p \in J^+(p)$.
(2): The *causal past* J^- and *chronological past* I^- are defined analogously.

Definition 5.3. A subset $S \subset M$ is called *achronal* if no two points in S are connected by a timelike curve.

Remark 5.4. Any edgeless spacelike hypersurface in Minkowski spacetime is achronal.

Definition 5.5. Given a closed achronal set S, we define the *future domain of dependence* of S as the set of all points p such that *every* past-moving inextendible causal curve through p must intersect S.

Remark 5.6. (1) Here inextendible means that the curve goes on forever, not ending at some finite point.
(2) Closed means that the complement are open sets.
(3) $S \subseteq D^+(S)$.
(4) The past domain of dependence $D^-(S)$ is defined by replacing the future with the past.

In general case, some points in our manifold M will be in one of the domains of dependence, and some will be outside. Now, we give the definition of *Cauchy horizon*

Definition 5.7. We define the *future Cauchy horizon* $H^+(S)$ to be the $\partial D^+(S)$. Similarly, $H^-(S)$ is defined to be the $\partial D^-(S)$.

Remark 5.8. Both of them are null surface.

6. Conformal diagram

At the very beginning, we mention that as $r \to \infty$, the Schwarzschild spacetime reduced to Minkowski spacetime. So in order to better understand the spacetime geometry, we want to pause a little and introduce conformal diagram. Let's begin with the Minkowski spacetime. The Minkowski metric in polar coordinates is

$$ds^2 = -dt^2 + dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)$$

Note that we can draw a light cones at $\frac{\pi}{4}$ everywhere ($t = \pm r$ are null). Here,

$$-\infty < t < \infty \qquad \text{and} \qquad 0 \leq r < \infty$$

Again, we do not want the range to be infinite. Using the same idea as before, we make a clever substitution to null coordinates:

$$\begin{cases} u = t - r \\ v = t + r \end{cases}$$

with the corresponding range given by

$$-\infty < u < \infty, \qquad -\infty < v < \infty \qquad \text{and} \qquad u \leq v$$

also the radius $r = \frac{1}{2}(v-u)$. Consequently, the Minkowski metric in null coordinates is given by

$$ds^2 = -\frac{1}{2}(dudv + dvdu) + \frac{1}{4}(v-u)^2(d\theta^2 + \sin^2\theta d\phi^2) \tag{6.1}$$

Now, we use the arctangent function to bring infinity into a finite coordinate value, letting

$$\begin{cases} U = \arctan u \\ V = \arctan v \end{cases}$$

which ranges

$$-\frac{\pi}{2} < U < \frac{\pi}{2}, \qquad -\frac{\pi}{2} < V < \frac{\pi}{2}, \qquad U \leq V$$

We then have

$$dudv + dvdu = \frac{1}{\cos^2 U \cos^2 V}(dUdV + dVdU)$$

and

$$\begin{aligned}(v-u)^2 = (\tan V - \tan U)^2 &= \frac{1}{\cos^2 U \cos^2 V}(\sin V \cos U - \cos V \sin U)^2 \\ &= \frac{1}{\cos^2 U \cos^2 V}\sin^2(V-U)\end{aligned}$$

Now, the metric (6.1) becomes

$$ds^2 = \frac{1}{4\cos^2 U \cos^2 V}\Big[-2(dUdV + dVdU) + \sin^2(V-U)(d\theta^2 + \sin^2\theta d\phi^2)\Big]$$

Now, we make a better transformation by transforming to a timelike coordinate T and a radial coordinate R, via

$$\begin{cases} T = V + U \\ R = V - U \end{cases}$$

with ranges

$$0 \leq R < \pi \qquad \text{and} \qquad |T| + R < \pi \tag{6.2}$$

Now, the metric is

$$ds^2 = \omega^{-2}(T,R)\big(-dT^2 + dR^2 + \sin^2 R\, d\Omega^2\big)$$

where

$$\begin{aligned}\omega &= 2\cos U \cos V \\ &= 2\cos\left[\frac{1}{2}(T-R)\right]\cos\left[\frac{1}{2}(T+R)\right] \\ &= \cos T + \cos R\end{aligned}$$

Now, we are almost done, the original Minkowski metric has been transformed into

$$\begin{aligned}\tilde{ds}^2 &= \omega^2(T,R)\, ds^2 \\ &= -dT^2 + dR^2 + \sin^2 R\, d\Omega^2\end{aligned}$$

Remark 6.1. This final form represents the manifold $\mathbf{R} \times S^3$, where the 3-sphere is purely spacelike, perfectly round, and unchanging in time. To see this, we just let

$$\begin{cases} x_0 = r\cos\psi \\ x_1 = r\sin\psi\cos\theta \\ x_2 = r\sin\psi\sin\theta\cos\phi \\ x_3 = r\sin\psi\sin\theta\sin\phi \end{cases} \implies ds^2 = r^2[d\psi^2 + \sin^2\psi(d\Omega^2)]$$

And these two match very well.

Here we look at the comparison pictures carefully.

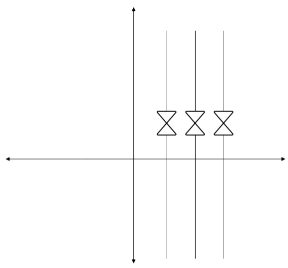

FIGURE 5. Timelike geodesic in $t - r$ coordinates

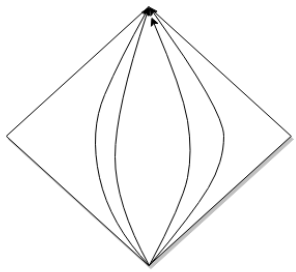

FIGURE 6. Timelike geodesic in conformal diagram

FIGURE 7. Comparison of timelike geodesic

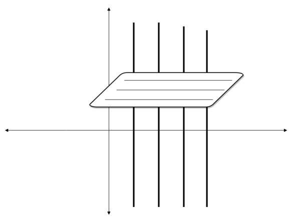

FIGURE 8. Spacelike geodesic in $t - r$ coordinates

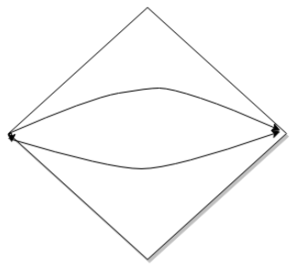

FIGURE 9. Sapcelike geodesic in conformal diagram

FIGURE 10. Comparison of Spacelike geodesic

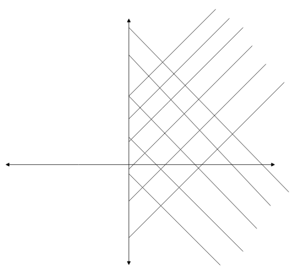

FIGURE 11. Null geodesic in $t - r$ coordinates

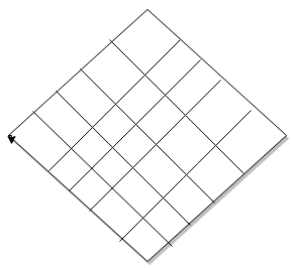

FIGURE 12. Null geodesic in conformal diagram

FIGURE 13. Comparison of Null geodesic

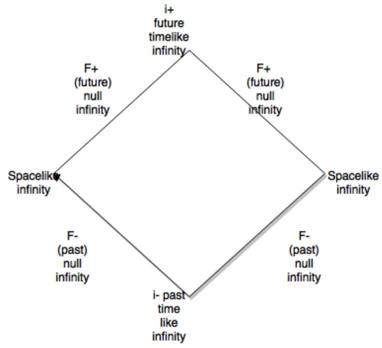

Figure 14. Conformal diagram

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