Review of Inflationary Cosmology

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Abstract

This paper is the term paper for Mathematical General Relativity course and a short pedagogical introduction to inflationary cosmology, highlighting selected areas of recent progress such as theory of cosmological perturbations.

I discuss Friedmann-Robertson-Walker cosmology and the horizon and flatness problems of the standard hot Big Bang, and introduce inflation as a solution to those problems, focusing on scenario of inflation from a single scalar field. The theory of cosmological perturbation provides the link between the models of very early universe and the data on the spectrum of density fluctuations and cosmic microwave anisotropies. And that is why is has become a cornerstone of modern cosmology. Here after a brief introduction, the classical and quantum theory of cosmological fluctuations is explained. Following a brief discussion about the current issues and the aspects of the theory which are still under investigation.

I. INTRODUCTION

The Cosmic Microwave Background (CMB) is the observational window which in recent years has yielded the most information. The anisotropies in the CMB have now been detected on a wide range of angular scales, giving us a picture of the Universe at the time of recombination.
The current data fits astonishingly well with the current paradigm of early Universe cosmology, the inflationary Universe scenario. However, it is important to keep in mind that what is tested observationally is the paradigm that the primordial spectrum of inhomogeneities was scale-invariant and predominantly adiabatic, and that there might exist other scenarios of the very early Universe which do not yield inflation but predict a scale-invariant adiabatic spectrum.

The theory of cosmological perturbations is what allows us to connect theories of the very early Universe with the data on the large-scale structure of the Universe at late times and is thus of central importance in modern cosmology.
the basic space-time diagram for inflationary cosmology is drawn below. Since, during the phase of standard cosmology $t_R < t < t_0$, where t_R corresponds to the end of inflation, and t_0 denotes the present time, the Hubble radius $l_H(t) \equiv \frac{1}{H(t)}$ expands faster that the physical wavelength associated with a fixed comoving scale, the wavelength becomes larger than the Hubble radius as we go backwards in time. However, during the phase of accelerated expansion (inflation), the physical wavelength increases much faster than the Hubble radius, and thus at early times the fluctuations emerged at micro-physical sub-Hubble scales. The idea is that micro-physical processes, quantum vacuum fluctuations are responsible for the origin of the fluctuations. However, during the period when the wavelength is super- Hubble, it is essential to describe the fluctuations using General Relativity. Thus, both Quantum Mechanics and General Relativity are required to successfully describe the generation and evolution of cosmological fluctuations.

II. STANDARD BIG BANG COSMOLOGY

1. *General Relativity and FRW Spacetime*

Contemporary cosmological models are based on the idea that the universe is very much the same everywhere-a stance sometimes known as the **Copernican principle**. This principle only applies on the very largest scales, where local variations in density are averaged over. Its validity on such scales is manifested in a number of different observations, most importantly in the 3K cos-

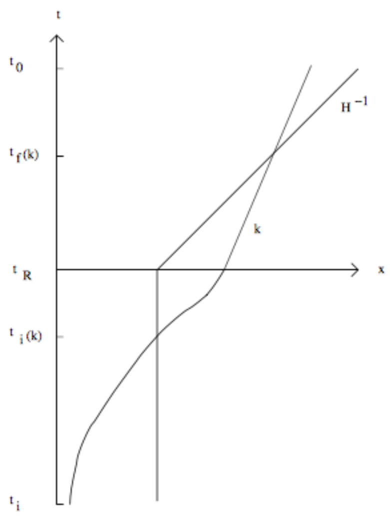

FIG. 1: Spacetime Diagram

mic microwave background (CMB). Although we know that the microwave background radiation is not perfectly smooth, the deviations from regularity are on the order of 10^{-5} or less, certainly an adequate basis for an approximate description of spacetime on large scales.

The Copernican principle is related to two more mathematically precise properties that a maniflod might have: isotropy and homogeneity.[6] A precise formulation can be given as follows: A spacetime is said to be (spatially) homogeneous if there exists a one-parameter family of spacelike hypersurfaces Σ_t foliating the spacetime (see Fig 2) such that for each t and for any points $p, q \in \Sigma_t$ there exists an isometry of the spacetime metric, $g_{\mu\nu}$, which takes p into q.

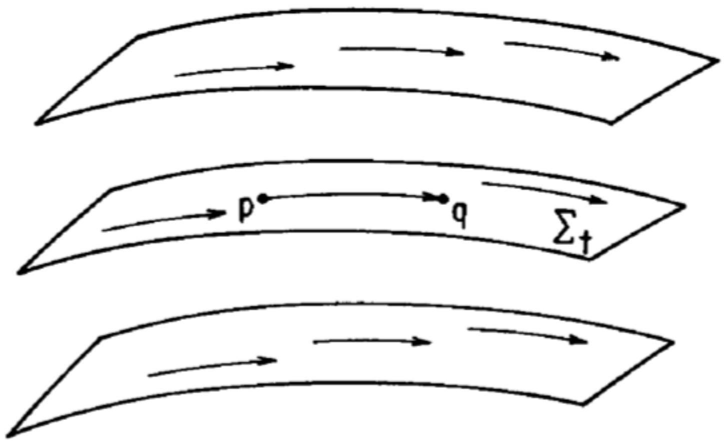

FIG. 2: The hypersurfaces of spatial homogeneity in spacetime

With regard to isotropy, it first should be pointed out that, in general, at each point, at most one observer can see the universe as isotropic . For example, if ordinary matter fills the universe, any observer in motion relative to the matter must see an anisotropic velocity distribution of the matter . With this fact in mind, a precise formulation of the notion of isotropy can be given as follows : A spacetime is said to be (spatially) isotropic at each point if there exists a congruence of timelike curves (i.e., observers), with tangents denoted u^{μ}, filling the spacetime (see Fig 3) and satisfying the following property . Given any point p and any two unit "spatial" tangent vectors $s_1^{\mu};, s_2^{\mu} \in V_p$ (i.e, vectors at p orthogonal to U^{μ}), there exists an isometry of $g_{\mu\nu}$ which leaves p and u^{μ} at p fixed but rotates s_1^{μ} ; into s_2^{μ} . Thus, in an isotropic universe it is impossible to construct a geometrically preferred tangent vector orthogonal to u^{μ}.

It is not difficult to see that in the case of a homogeneous and isotropic spacetime, the surfaces Σ_t

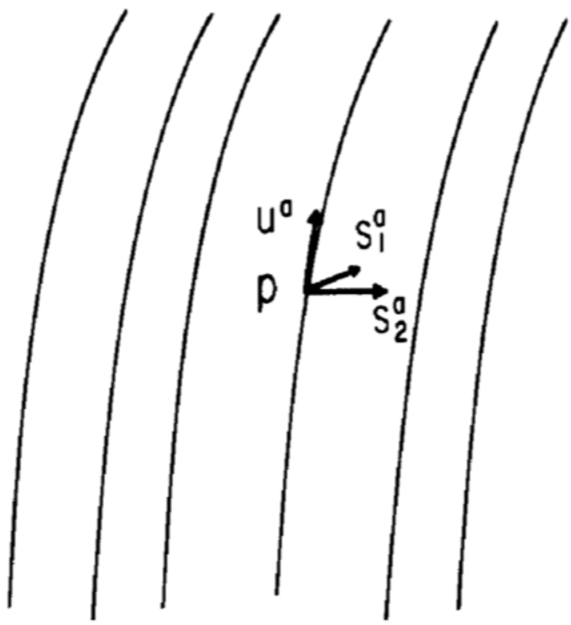

FIG. 3: The world lines of isotropic observers in spacetime

of homogeneity must be orthogonal to the tangents, u^{μ}, to the world lines of the isotropic observers. If not, then assuming that the isotropic observers and the family of homogeneous surfaces Σ_t are unique, the failure of the tangent subspace orthogonal to u^{μ} to coincide with the tangent space of Σ_t would enable us to construct a geometrically preferred spatial vector, in violation of isotropy. Using these two properties it can be shown that the curvature must be constant. Also it was proved by Eisenhart that any two spaces of constant curvature of the same dimension and metric signature which have equal values of K must be (locally) isometric . Thus, our task of determining the possible spatial geometries of Σ_t will be completed if we enumerate spaces of constant curvature encompassing all values of K. This is easily done. And the result is given [1]

$$ds^2 = -dt^2 + a^2(t)\left(\frac{dr^2}{1-kr^2} + r^2(d\theta^2 + sin^2\theta d\phi^2)\right) \tag{1}$$

Here, the scale factor a(t) characterizes the relative size of spacelike hypersurfaces Σ_t. The curvature parameter k is +1 for positively curved Σ, 0 for flat Σ, and 1 for negatively curved Σ. Above equation uses comoving coordinates the universe expands as a(t) increases, but galaxies/ observers keep fixed coordinates r, θ, ϕ as long as there are not any forces acting on them, i.e. in the absence of peculiar motion. The corresponding physical distance is obtained by multiplying with the scale factor, R = a(t)r, and is time-dependent even for objects with vanishing peculiar velocities.The metric is an important concept in General Relativity. However one should notice in General Relativity distribution of mass/energy in the spacetime determines the shape of the metric, and the metric in turn determines evolution of mass/energy.

$$G_{\mu\nu} = 8\pi G T_{\mu\nu} \tag{2}$$

The tensor $G_{\mu\nu}$is a symmetric 4×4 tensor consisting of the metric,and its first and second derivatives. The Einstein Field Equation therefore represents a set of ten coupled nonlinear, second order partial differential equations of ten free fictions which are the elements of the metric tensor. However only six of these equations are actually independent, leaving four degree of freedom. The physics of gravity is independent of coordinate system, and the additional degree of freedom correspond to a choice of coordinate system, or gauge on the four dimensional space. Also the most general homogeneous, isotropic stress-energy is diagonal, with all of its spatial component identical,

$$T^{\mu}_{\nu} = \begin{pmatrix} \rho(t) & 0 & 0 & 0 \\ 0 & -p(t) & 0 & 0 \\ 0 & 0 & -p(t) & 0 \\ 0 & 0 & 0 & -p(t) \end{pmatrix} \tag{3}$$

Where we identify the energy density ρ and the pressure p from continuity equation arising from stress energy conservation,

$$T^{\mu\nu}_{;\nu} = \dot{\rho} + 3\left(\frac{\dot{a}}{a}\right)(\rho + p) = 0 \tag{4}$$

The Einstein field equations then reduce to a set of two coupled,nonlinear ordinary differential equations,

$$\left(\frac{\dot{a}}{a}\right)^2 + \frac{k}{a^2} = \frac{8\pi}{3m_{pl}^2}\rho$$
$$\left(\frac{\ddot{a}}{a}\right) = -\frac{4\pi}{3m_{pl}^2}(\rho + 3p) \tag{5}$$

The first is called the Friedmann equation and the second is called the Raychaudhurri equation. The expansion rate $\frac{\dot{a}}{a}$ is called the Hubble parameter.

Any particle moving in an expanding FRW spacetime will lose momentum as $p \propto a^{-1}$. For massless particles like photons, this is manifest as redshift in the wavelength, but it means that a massive particle will asymptotically come to rest relative to the comoving coordinate system. thus comoving coordinates represent a preferred reference frame: any free body with peculiar velocity relative to the comoving frame will eventually come to rest in that frame. For further references we notice that Ω defined as the ratio of the actual density ρ to the critical density $\rho_c \equiv \frac{3m_{Pl}^2}{8\pi}H^2 \rightarrow k = 0$(corresponding to a geometrically flat universe) using Friedmann equation can be written as:

$$\Omega(t) = 1 + \frac{k}{(aH)^2} \tag{6}$$

The key assumption of standard cosmology is that matter is described by a classical ideal gas with an equation of state:

$$p = \omega\rho \tag{7}$$

For cold matter pressure is negligible so $\omega = 0$ and it follows:

$$\rho_m(t) \sim a^{-3}(t) \tag{8}$$

For radiation we have $\omega = \frac{1}{3}$ and hence it follows:

$$\rho_r(t) \sim a^{-4}(t) \tag{9}$$

If we take the stress-energy and add a term proportional to the metric the identity $D_\nu g^{\mu\nu} = 0$ means the stress energy conservation is unchanged. This constant does not affect local dynamics and corresponds to an equation of state:

$$p_\Lambda = -\rho_\Lambda \tag{10}$$

and $\omega = -1$ and

$$\rho_\Lambda = Const \tag{11}$$

This is the third type of stress-energy.

The epoch at which atom forms, when the universe was at the age of 300,000 years and at the temperature 3000 k is called recombination, before that universe was an ionized plasma of mostly protons, electrons, and photons. The important characteristic of this plasma was that it was opaque the mean free path was very smaller than Hubble length. After the neutralization of matter in universe, photons propagate with black-body distribution of frequencies with the background radiation temperature T=3000. We can detect these photons now. By using equation $T \propto a(t)^{-1}$ we can determine the redshift of recombination:

$$1 + z_R = \frac{a(t_0)}{a(t_R)} = \frac{T_R}{T_0} \simeq 1100 \tag{12}$$

This is the cosmic microwave background. we can view the observation of the CMB photons as imaging a uniform surface of last scattering at redshift 1100. The observed CMB is highly isotropic but it is not perfectly so. This anisotropy represents intrinsic fluctuation in the CMB itself, due to the presence of tiny primordial density fluctuations in the cosmological matter present at the time of recombination. These are the fluctuations which later collapsed to form all of the structure in the universe. The simplest contribution to the CMB anisotropy from density fluctuations is gravitational redshift. A photon coming from a region which is slightly denser than the average will have a lightly larger redshift due to the deeper gravitational well at the surface of last scattering. This contribution is dominant on large angular scales. For fluctuation modes on smaller angular scales the dominant process is acoustic oscillation in the baryon/photon plasma. Matter tends to collapse due to gravity onto regions where the density is higher than average, so the baryons fall into over-dense regions. However, since the baryons and photons are still strongly coupled, the photons tend to resist this collapse and push the baryons outward. The result is oscillatory modes of compression and rarefaction in the gas due to density fluctuations. The gas heats as it compresses and cools as it expands, which creates fluctuations in the temperature of the CMB.[4]

III. BIG BANG PUZZLES

The comoving region $l_p(t_{rec})$ over which the CMB is observed to be homogeneous to better than one part in 10^4 is much larger than the comoving forward light cone $l_f(t_{rec})$ at t_{rec}, which is the maximal distance over which micro-physical could have caused the homogeneity:

$$l_p(t_{rec}) = \int_{t_{rec}}^{t_0} dt a^{-1}(t) \simeq 3t_0 \left(1 - \left(\frac{t_{rec}}{t_0}\right)^{\frac{1}{3}}\right)$$
$$l_f(t_{rec}) = \int_0^{t_{rec}} dt a^{-1}(t) \simeq 3t_0^{\frac{2}{3}} t_{rec}^{\frac{1}{3}} \tag{13}$$

From the above equations it is obvious that $l_p(t_{rec}) \gg l_f(t_{rec})$. Hence standard cosmology can not explain the observed isotropy of the CMB. This is the horizon problem.
Also, by using Friedmann equations we can show the evolution the density parameter is:

$$\frac{d\Omega}{dlna} = (1 + 3\omega)\Omega(\Omega - 1) \tag{14}$$

For which by inputting different values of ω for matter and radiation we get that $\Omega = 1$ is an unstable fixed point. So any deviation from flat geometry is amplified by subsequent cosmological expansion, so a nearly flat universe today is a highly fine-tuned situation. So why did universe start out so incredibly close to flat? We call this the flatness problem. The third problems is the formation of structure. Observations indicate that galaxies and even clusters of galaxies have nonrandom correlations on scales larger than 50 Mpc. The questions of what generates the primordial density perturbations and what causes the observed correlations do not have an answer in the context of standard cosmology.

IV. INFLATIONARY COSMOLOGY

The idea of inflation is very simple. We assume there is a time interval beginning at t_i and ending at t_R (the reheating time) during which the Universe is exponentially expanding, i.e.,

$$a(t) \sim e^{Ht}, \quad t \in [t_i, t_R] \tag{15}$$

with constant Hubble expansion parameter H. Such a period is called de Sitter or inflationary. The success of Big Bang nucleosynthesis sets an upper limit to the time t_R of reheating:

$$t_R \ll t_{NS} \tag{16}$$

t_{NS} being the time of nucleosynthesis.
Fig. 4 is a sketch of how a period of inflation can solve the homogeneity problem. $\Delta t = t_R t_i$ is the period of inflation. During inflation, the forward light cone increases exponentially compared to a model without inflation, whereas the past light cone is not affected for $t \geq t_R$. Hence, provided

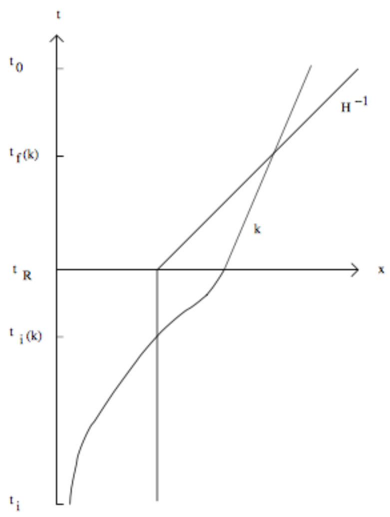

FIG. 4: Spacetime Diagram

Δt is sufficiently large, $l_f(t_R)$ will be greater than $l_p(t_R)$. Inflation also can solve the flatness problem. The key point is that the entropy density s is no longer constant. As will be explained later, the temperatures at t_i and t_R are essentially equal. Hence, the entropy increases during inflation by a factor $e^{(3Ht)}$. Thus, ϵ decreases by a factor of $e^{(2Ht)}$. Hence, ρ and ρ_C can be of comparable magnitude at both t_i and the present time. In fact, if inflation occurs at all, then rather generically, the theory predicts that at the present time $\Omega = 1$ to a high accuracy (now $\Omega < 1$ requires special initial conditions or rather special models). Most importantly, inflation provides a causal mechanism for generating the primordial perturbations required for galaxies, clusters and even larger objects. In inflationary Universe models, the Hubble radius (apparent horizon), 3t, and the (actual) horizon (the forward light cone) do not coincide at late times. Provided that the duration of inflation is sufficiently long, then all scales within our present apparent horizon were inside the horizon since t_i. Thus, in principle it is possible to have a casual generation mechanism for perturbations. The generation of perturbations is supposed to be due to a causal micro-physical process. Such processes can only act coherently on length scales smaller than the Hubble radius $l_H(t)$, where

$$l_H(t) = H^{-1}(t) \tag{17}$$

A heuristic way to understand $l_H(t)$ is to realize that it is the distance which light (and hence the maximal distance any causal effects) can propagate in one expansion time.

V. HOW TO OBTAIN INFLATION

Obviously, the key question is how to obtain inflation. From the FRW equations, it follows that in order to get an exponential increase of the scale factor, the equation of state of matter must be

$$p = -\rho \tag{18}$$

which is not compatible with the standard (cosmological) model description of matter as an ideal gas of classical matter. As mentioned earlier, the ideal gas description of matter breaks down in the very early Universe. Matter must, instead, be described in terms of quantum field theory (QFT). In the resulting framework (classical general relativity as a description of space and time, and QFT as a description of the matter content) it is possible to obtain inflation. More important than the quantum nature of matter is its field nature. Note, however, that quantum field driven inflation is not the only way to obtain inflation. In fact before the seminal paper by Guth, Starobinsky proposed a model with exponential expansion of the scale factor based on higher derivative curvature terms in the gravitational action. Current quantum field theories of matter contain three types of fields: spin $\frac{1}{2}$ fermions (the matter fields) ψ, spin 1 bosons A_μ (the gauge bosons) and spin 0 bosons, the scalar fields ϕ (the Higgs fields used to spontaneously break internal gauge symmetries). The Lagrangian of the field theory is constrained by gauge invariance, minimal coupling and renormalizability. The Lagrangian of the bosonic sector of the theory is thus constrained to have the form

$$L_m(\phi, A_\mu) = \frac{1}{2}D_\mu\phi D^\mu\phi - V(\phi) + \frac{1}{4}F_{\mu\nu}F^{\mu\nu} \tag{19}$$

where D_μ denotes the (gauge) covariant derivative, g being the gauge coupling constant, $F_{\mu\nu}$ is the field strength tensor, and $V(\phi)$ is the Higgs potential. Renormalizability plus assuming symmetry under $\phi \to -\phi$ constrains $V(\phi)$ to have the form

$$V(\phi) = \frac{1}{2}m^2\phi^2 + \frac{1}{4}\lambda\phi^4 \tag{20}$$

where m is the mass of the excitations of ϕ about $\phi = 0$, and λ is a self- coupling constant. For spontaneous symmetry breaking, $m^2 < 0$ is required. Given the Lagrangian (21), the action for matter is

$$S_m = \int d^4x\sqrt{-g}\mathcal{L}_m \tag{21}$$

where g here denotes the determinant of the metric tensor, and now the covariant derivative D_μ in (21) is a gauge and metric covariant derivative. The energy-momentum tensor is obtained by varying this action with respect to the metric. The contributions of the scalar fields to the energy density ρ and pressure p are

$$\begin{aligned} \rho(\phi) &= \frac{1}{2}\dot{\phi}^2 + \frac{1}{2}a^{-2}(\nabla\phi)^2 + V(\phi) \\ p(\phi) &= \frac{1}{2}\dot{\phi}^2 - \frac{1}{2}a^{-2}(\nabla\phi)^2 - V(\phi) \end{aligned} \tag{22}$$

It thus follows that if the scalar field is homogeneous and static, but the potential energy positive, then the equation of state $p = \rho$ necessary for exponential inflation results. This is the idea behind potential-driven inflation. Note that given the restrictions imposed by minimal coupling, gauge invariance and renormalizability, scalar fields with nonvanishing potentials are required in order to obtain inflation. Mass terms for fermionic and gauge fields are not compatible with gauge invariance, and renormalizability forbids nontrivial potentials for fermionic fields. Most of the current realizations of potential-driven inflation are based on satisfying the conditions

$$\dot{\phi}^2, a^{-2}(\nabla\phi)^2 \ll V(\phi) \tag{23}$$

via the idea of slow rolling.[3]

VI. RELATIVISTIC THEORY OF COSMOLOGICAL FLUCTUATIONS

A. Introduction

Observations of the Cosmic Microwave background explain the success of cosmological perturbation theory. At the time of decoupling the universe was very homogeneous with small inhomogeneities at the 10^{-5} level. Of course the natural strategy in is this case is to split all quantities, metric and matter fields into a homogeneous background that depend only on time and fluctuations which are function of space and time.[2] The metric of a homogeneous isotropic background is is FRW metric which can be written in conformal time η (defined $dt = a(t)d\eta$) as:

$$ds^2 = a(\eta)^2(d\eta^2 - dx^2) \tag{24}$$

The Einstein Equations which took the form of two coupled, non-linear differential equations, are

$$H^2 \equiv \left(\frac{\dot{a}}{a}\right) = \frac{1}{3}\rho - \frac{k}{a^2} \tag{25}$$

and

$$\dot{H} + H^2 = -\frac{1}{6}(\rho + 3p) \tag{26}$$

Combining these two equations together:

$$\frac{d\rho}{dt} + 3H(\rho + p) = 0 \tag{27}$$

Which determine the expansion rate and its time derivative in terms of the equation of state of the matter, whose background stress-energy was written as:

$$T^{\mu}_{\nu} = \begin{pmatrix} \rho & 0 & 0 & 0 \\ 0 & -p & 0 & 0 \\ 0 & 0 & -p & 0 \\ 0 & 0 & 0 & -p \end{pmatrix} \tag{28}$$

The theory of cosmological perturbations is based on expanding the Einstein equations to linear order about the background metric.

B. Classifying Fluctuations

The spatially flat homogeneous and isotropic background spacetime possesses a great deal of symmetry. These symmetries allow a decomposition of the metric and the stress-energy perturbations into independent Scalar (S), vector (V) and tensor (T) components. This SVT decomposition classify different component according to their transformation properties under spatial rotations and is most easily described in Fourier space

$$X_k(t) == \int d^3x X(t,x) e^{ik.x} \qquad X \equiv \delta\phi, \delta g_{\mu\nu}, etc. \tag{29}$$

The translation invariance of the linear equations of motion for the perturbations means that the different Fourier modes do not interact. Different Fourier modes can therefore be studied independently. This often simplifies the differential equations for the perturbations. Now we consider rotations around a single Fourier wavevector k. A perturbation is said to have helicity m if its amplitude is multiplied by $e^{im\psi}X_k$ under rotation of the coordinate system around the wavevector by an angle ψ.

$$X_k \to e^{im\psi} X_k \tag{30}$$

Scalar, vector and tensor perturbations have helicity 0, ± 1 and ± 2, respectively. The importance of the SVT decomposition is that the perturbations of each type evolve independently (at the linear level) and can therefore be treated separately.[2] Now by expanding the metric about the FLRW bacground metric $g^{(0)}_{\mu\nu}$ given by:

$$g_{\mu\nu} = g^{(0)}_{\mu\nu} + \delta g_{\mu\nu} \tag{31}$$

By SVT decomposition we can see that there are four degrees of freedom which correspond to scalar metric fluctuations(the only four ways of constructing a metric from scalar functions)

$$\delta g_{\mu\nu} = a^2 \begin{pmatrix} 2\phi & -B_{,i} \\ -B_{,i} & 2(\psi\delta_{ij} - E, ij) \end{pmatrix} \tag{32}$$

Where the four fluctuating degrees of freedom are denoted.

There are also four vector degrees of freedom of metric fluctuations, which are the four ways of constructing metric fluctuations from three vectors:

$$\delta g_{\mu\nu} = a^2 \begin{pmatrix} 0 & -S_i \\ -S_i & F_{i,j} + Fj, i \end{pmatrix} \tag{33}$$

Where S_i and F_i are two divergenceless vectors. Finally there are two tensor modes which correspond to the two polarization states of gravitational waves which do not couple at linear order to the matter fluctuations. So they are not important for our purpose of study. Vector fluctuations decay in an expanding background cosmology and hence are not usually cosmologically important. The most important fluctuations, at least in inflationary cosmology, are the scalar metric fluctuations, the fluctuations which couple to matter inhomogeneities.[5]

C. Gauge Choice

A crucial subtlety in the study of cosmological perturbations is the fact that the split into background and perturbations, is not unique, but depends on the choice of coordinates or the gauge choice. it is important to realize that the slicing and threading of the perturbed spacetime is not unique. Furthermore, when describing an inhomogeneous spacetime there is often not a preferred coordinate choice. When we make a gauge choice to define the slicing and threading of the spacetime we implicitly also define the perturbations. To demonstrate this fact, consider an unperturbed homogeneous and isotropic universe, where the energy density is only a function of time, $\rho(t, x) = \rho(t)$. We now show that a change of the time coordinate can introduce fictitious perturbations $\delta\rho$. Consider a new time coordinate $\tilde{t} = t + \delta t(t, x)$. In general, the energy density on the new time-slice will not be homogeneous, $\tilde{\rho}(\tilde{t}, x) = \rho(t(\tilde{t}, x))$. These perturbations in the energy density arent physical, but entirely due to the choice of new time-slicing. Similarly, we can remove a real perturbation in the energy density by choosing the hypersurface of constant time to coincide with the hypersurface of constant energy density. Then $\delta\tilde{\rho} = 0$ although there are real inhomogeneities. To resolve ambiguities between real and fake perturbations in general relativity, we need to consider the complete set of perturbations, i.e. we need both the matter field perturbations and the metric perturbations and by a gauge transformation we can trade one for the other. To avoid misinterpretation of fictitious gauge modes it will also be useful to study gauge-invariant combinations of perturbations. By definition, fluctuations of gauge-invariant quantities cannot be removed by a coordinate transformation.[2]

VII. EQUATION OF MOTION

We begin with the Einstein equations

$$G_{\mu\nu} = 8\pi G T_{\mu\nu} \tag{34}$$

insert the ansatz for metric and matter perturbed about a FRW background.

$$\begin{aligned} g_{\mu\nu}(x,\eta) &= g^{(0)}_{\mu\nu}(\eta) + \delta g_{\mu\nu}(x,\eta) \\ \phi(x,\eta) &= \phi_0(\eta) + \delta\phi(x,\eta) \end{aligned} \tag{35}$$

Notice that we are only considering scalar matter field. Expanding to linear order in the fluctuating fields, generates the following equations:

$$\delta G_{\mu\nu} = 8\pi G \delta T_{\mu\nu} \tag{36}$$

Also note that the components $\delta G_{\mu\nu}$ and $\delta T_{\mu\nu}$ are not gauge invariant. If we want to use the gauge-invariant approach, we note that it is possible to construct a gauge-invariant tensor $\delta G_{\nu}^{(gi)\mu}$. In terms of these tensors, the gauge-invariant form of the equations of motion for linear fluctuations reads

$$\delta G^{(gi)}_{\mu\nu} = 8\pi G \delta T_{\mu\nu} \tag{37}$$

Now by restricting our attention to the case of matter described in terms of a single scalar field ϕ with action

$$S = \int d^4x \sqrt{-g} \left[\frac{1}{2}\phi^{,\alpha}\phi_{,\alpha} - V(\phi) \right] \tag{38}$$

After some manipulation we can write the following second order differential equation for the relativistic potential ϕ:

$$\phi'' + 2\left(\mathcal{H} - \frac{\phi_0''}{\phi_0'}\right)\phi' - \nabla^2\phi + 2\left(\mathcal{H}' - \mathcal{H}\frac{\phi_0''}{\phi_0'}\right)\phi = 0 \tag{39}$$

To study the quantitative implications of the equation of motion (39), it is convenient to introduce the variable ζ (which, up to correction terms which are unimportant for large-scale fluctuations) is equal to the curvature perturbation R in comoving gauge.

$$\zeta \equiv \phi + \frac{2}{3}\frac{(H^{-1}\dot{\phi} + \phi)}{1+\omega} \tag{40}$$

Where

$$\omega = \frac{p}{\rho} \tag{41}$$

On large scales, we have:

$$\dot{\zeta}(1+\omega) = 0 \tag{42}$$

This implies that except if $1 + \omega = 0$ at some points in time during cosmological evolution ζ is constant. In single matter field models it is indeed possible to show that ζ on super-Hubble scales is independent of assumptions on the equation of state. This conservation law makes it easy to relate initial fluctuations to final fluctuations in inflationary cosmology. After some manipulation we get to the final result which is:

$$\phi_{(t_f(k))} \sim \frac{V^{\frac{3}{2}}}{V'}(t_i(k)) \tag{43}$$

which gives the position space amplitude of cosmological fluctuations on a scale labelled by the comoving wavenumber k at the time when the scale re-enters the Hubble radius at late times. In the case of slow roll inflation, the right hand side of (43) is, to a first approximation, independent of k, and hence the resulting spectrum of fluctuations is scale-invariant.

VIII. QUANTUM THEORY OF COSMOLOGICAL FLUCTUATIONS

From the classical theory of cosmological perturbations discussed in the previous section, we know that the analysis of scalar metric inhomogeneities can be reduced - after extracting gauge artifacts - to the study of the evolution of a single fluctuating variable. Thus, we conclude that the quantum theory of cosmological perturbations must be reducible to the quantum theory of a single free scalar field which we will denote by v. We begin with the Einstein- Hilbert action and action of a scalar matter field, by fixing the gauge to be longitudinal gauge and by doing a long calculation we can write the result which is a contribution $S^{(2)}$ to the action quadratic in the perturbations. And then the equation of motion in the momentum space would follow to be:

$$v_k'' + k^2 v_k - \frac{z''}{z} v_k = 0 \tag{44}$$

where v_k is the kth Fourier mode of v and v is a function of ϕ and $\delta\phi$. And z in the case of both power law inflation and slow roll inflation is proportional to a:

$$z(\eta) \sim a(\eta) \tag{45}$$

it immediately follows that on small length scales, i.e. for $k > k_H$, the solutions for v_k are constant amplitude oscillations . These oscillations freeze out at Hubble radius crossing, i.e. when $k = k_H$. On longer scales ($k \ll k_H$), the solutions for v_k increase as z.[5]

$$v_k \sim z \qquad k \ll k_H \tag{46}$$

To summarize the quantum theory of cosmological perturbations, we can say in the linearized theory, fluctuations are set up at some initial time t_i mode by mode in their vacuum state. While the wavelength is smaller than the Hubble radius, the state undergoes quantum vacuum fluctuations. The accelerated expansion of the background redshifts the length scale beyond the Hubble radius. The fluctuations freeze out when the length scale is equal to the Hubble radius. On larger scales, the amplitude of v_k increases as the scale factor. This corresponds to the squeezing of the quantum state present at Hubble radius crossing (in terms of classical general relativity, it is self- gravity which leads to this growth of fluctuations). The squeezing of the quantum vacuum state leads to the emergence of the classical nature of the fluctuations.
In inflationary cosmology we can compute the power spectrum $\mathcal{P}_{\mathcal{R}}(k)$ of the curvature fluctuation $\mathcal{R}$, since $\mathcal{R}$ in comoving coordinate is related to v. and we get a scale invariant power spectrum with amplitude proportional to H^2, in agreement with what was discussed in the last section.

IX. THE TRANS-PLANCKIAN WINDOW

In this section and the following section we deal with aspects of the theory of cosmological perturbations which are currently under investigation and are controversial.
The same background dynamics which yields the causal generation mechanism for cosmological fluctuations, the most spectacular success of inflationary cosmology, bears in it the nucleus of the trans-Planckian problem. If inflation lasts only slightly longer than the minimal time it needs to last in order to solve the horizon problem and to provide a causal generation mechanism for CMB fluctuations, then the corresponding physical wavelength of these fluctuations is smaller than the Planck length at the beginning of the period of inflation. The theory of cosmological perturbations is based on classical general relativity coupled to a weakly coupled scalar field description of matter.

Both the theories of gravity and of matter will break down on trans-Planckian scales, and this immediately leads to the trans-Planckian problem: are the predictions of standard inflationary cosmology robust against effects of trans-Planckian physic.

X. BACK-REACTION OF COSMOLOGICAL FLUCTUATIONS

The presence of cosmological fluctuations influences the background cosmology in which the perturbations evolve. This back-reaction arises as a second order effect in the cosmological perturbation expansion. The effect is cumulative in the sense that all fluctuation modes contribute to the change in the background geometry, and as a consequence the back-reaction effect can be large even if the amplitude of the fluctuation spectrum is small. To quantify back- reaction, the effect of the fluctuations on the background is expressed in terms of an effective energy-momentum tensor. It can be shown that in the context of an inflationary background cosmology, the long wavelength contributions to the effective energy-momentum tensor take the form of a negative cosmological constant, whose absolute value increases as a function of time since the phase space of infrared modes is increasing. This then leads to the speculation that gravitational back-reaction may lead to a dynamical cancellation mechanism for a bare cosmological constant, and yield a scaling fixed point in the asymptotic future in which the remnant cosmological constant satisfies $\Omega_\Lambda \sim 1$. Also one can find that the leading infrared back-reaction contributions cancel in single field inflationary models. However, we expect non-trivial back-reaction of infrared modes in models with more than one matter field. However, there are important concerns about the above formalism, and even more so about the resulting speculations. On a formal level, since our back-reaction effect is of second order in cosmological perturbation theory, it is necessary to demonstrate covariance of the proposed back-reaction equation beyond linear order, and this has not been done. Next, it might be argued that by causality super-Hubble fluctuations cannot affect local observables. Finally, from an observational perspective one is not interested in the effect of fluctuations on the background metric (since what the background is cannot be determined precisely using local observations). Instead, one should compute the back-reaction of cosmological fluctuations on observables describing the local Hubble expansion rate. One might then argue that even if long-wavelength fluctuations have an effect on the background metric, they do not influence local observables.

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